Exact analytical expressions for the piezometric profile and water exchange between stream and groundwater during and after a uniform rise of the stream level

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[1] We consider an archetypical problem relevant to a confined aquifer in contact with a stream. The model problem consists of an idealized one-dimensional region $0 \leq x \leq \mathrm{L}$, where the left boundary at $x = 0$ is held at a fixed piezometric head h_0, and the right boundary's piezometric head at $x = L$ is increased from h_L to h_0 at a constant rate. Exact solutions for all times, all points in the aquifer, and for any possible constant rate of change of the right boundary piezometric head are presented for the piezometric head and the instantaneous flow rate. An exact expression for the exchange volume at the groundwater/stream interface for an arbitrary time is also provided. This expression shows that there is a specific critical rising rate of the stream level above which the net exchange volume is into the aquifer and below which it is out of the aquifer. The solution shows that regardless of the rise rate, a certain water volume, inversely proportional to the rise rate, enters the aquifer. *INDEX TERMS:* 1829 Hydrology: Groundwater hydrology; 1836 Hydrology: Hydrologic budget (1655); 1831 Hydrology: Groundwater quality; *KEYWORDS:* storage, diffusion, surface water, groundwater, biochemical, exchange

1. Introduction

[2] Quantifying water exchange between the groundwater and an open water body in transient regime is an important task in many environmental and water resources applications. Examples include solute transport in coastal aquifers subjected to tide [see *Venosa et al.*, 1996; *Boufadel et al.*, 1999a; *Boufadel*, 2000, and references therein] and hyporheic exchange in groundwater/stream systems during flood events [*Mulholland*, 1992; *Lockaby and Conner*, 1999].

[3] The quantification is a challenging task because of, among other factors, unsaturated flow hydraulics [*Boufadel et al.*, 1998, 1999a, 1999b; *Boufadel*, 2000] and subsurface heterogeneity [e.g., *Molz et al.*, 1997; *Boufadel et al.*, 2000].

[4] This work investigates a simplified situation (Figure 1) where a homogeneous confined layer is in contact with an open water body (henceforth referred to as "stream") whose level is rising at a constant speed. The system in Figure 1 represents a practical situation where a high permeability layer is confined between two impermeable (or very low permeability) layers. We develop closed-form exact analytical expressions for the piezometric profile and the water flow in the confined layer. Such expressions could be used as test cases for numerical models and guidelines for experimental works.

[5] The initial piezometric profile in Figure 1 is linear, and groundwater flows toward the stream. The water level in the stream is then raised at a constant speed within a time t_r (the "r" is for "rise") from h_L to h_0, where it is kept there indefinitely ($t \rightarrow \infty$). We show in this work that the speed of rise (i.e., the value of t_r) greatly affects the exchange volume between the stream and groundwater. In particular, we show that a fast rising stream level results in stream water entering the subsurface in spite of the fact that the stream level is always $\leq h_0$. This mechanism is known as bank storage [*Pinder and Sauer*, 1971; *Hunt*, 1990].

2. Problem Statement

[6] The governing equation for one-dimensional water flow in saturated homogeneous isotropic porous media is

$$\frac{\partial h}{\partial t} = \alpha \frac{\partial^2 h}{\partial x^2}, \tag{1}$$

where h is the water head [L], $\alpha\,[L^2T^{-1}]$ is the aquifer diffusivity, $\alpha = S_0/K$, where $S_0\,[L^{-1}]$ is the specific storativity, and $K\,[LT^{-1}]$ is the (saturated) hydraulic conductivity [*Bear*, 1988]. The initial condition is (Figure 1)

$$h(x, t=0) = h_0 + (x/L)(h_L - h_0). \tag{2}$$

The boundary conditions are (Figure 1) as follows for left side, $x = 0$,

$$h(x=0, t) = h_0 \tag{3a}$$

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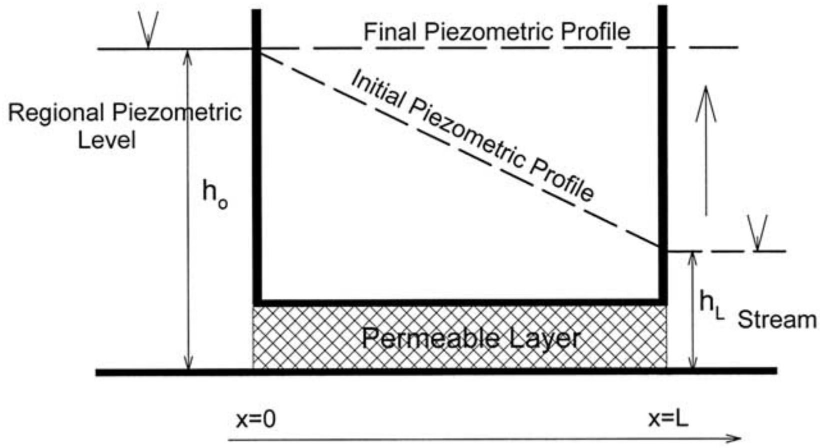

Figure 1. Specification of the problem analyzed in this work. The water level in the stream starts from h_L and is raised uniformly during a time t_r until reaching h_0. It is then left there for an infinite time.

and for right side, $x = L$,

$$h(x = L, t) = h_L + (t/t_r)(h_0 - h_L), \quad t < t_r \tag{3b}$$

$$h(x = L, t) = h_0, \quad t \geq t_r, \tag{3c}$$

where t_r is time for $h\ (L, t)$ to rise from h_L to h_0.

[7] Equations (3b) and (3c) state that the right-side boundary (i.e., the stream level) starts to vary from h_L at time $t = 0$ and reaches its final state h_0 at time $t = t_r$. It is shown in this work that it would take longer than t_r for the piezometric profile within the domain to reach its final steady state value $h = h_0$. Existing analytical solutions are either approximate or obtained in the limiting situations where the rise period, t_r, is very small ($t_r \ll 1$, a sudden increase) or very large ($t_r \rightarrow \infty$, an infinitely slow rise) [*Polubarinova-Kochina*, 1962; *Carslaw and Jaeger*, 1954; *Bear*, 1988; *Barenblatt et al.*, 1990; *Barlow and Moench*, 2000]. A solution when the rise follows an asymmetric cosine curve was presented by *Cooper and Rorabaugh* [1963]. We present below an exact closed-form analytical solution for the problem given by equations (1)–(3).

[8] It is both convenient and theoretically valuable to pose the problem in dimensionless variables, defined here as

$$\xi \equiv x/L, \tag{4a}$$

$$\tau \equiv \frac{\alpha t}{L^2}, \tag{4b}$$

$$\theta \equiv \frac{h - h_0}{h_L - h_0}. \tag{4c}$$

Notice that θ is of opposite sign to h because $(h_L - h_0)$ is negative. The problem is restated in a dimensionless form as

$$\frac{\partial^2\theta(\xi,\tau)}{\partial\xi^2} = \frac{\partial\theta(\xi,\tau)}{\partial\tau}. \tag{5}$$

Initial condition is

$$\theta(\xi, \tau = 0) = \xi. \tag{6}$$

Boundary conditions are

$$\text{Left boundary} \quad \theta(\xi, = 0, \tau) = 0 \tag{7a}$$

$$\text{Right boundary} \quad \theta(\xi = 1, \tau) = 1 - \tau/\gamma, \quad \tau < \gamma \tag{7b}$$

$$\theta(\xi = 1, \tau) = 0, \quad \tau \geq \gamma, \tag{7c}$$

where

$$\gamma \equiv \frac{\alpha t_r}{L^2}. \tag{8}$$

The parameter γ represents the period of rise normalized by the characteristic time L^2/α that depends on aquifer properties and geometry. This parameter is the important governing parameter in the problem under study. Note that all terms in equations (5)–(8) are dimensionless and that θ decreases with time at the right-side boundary.

3. Analytical Solution

[9] An exact solution for the problem stated in equations (5)–(8) may be obtained as follows. First, the entire problem statement is transformed from a partial differential equation (PDE) to an ordinary differential equation (ODE) using the Laplace transform [*Brown and Churchill*, 1996; *Kreyszig*, 1999]. The Laplace transform of the function $\theta(\xi, \tau)$ is denoted by $\Theta(\xi, s)$ and is given by

$$\Theta(\xi, s) = L(\theta(\xi, \tau)) = \int_0^\infty \theta(\xi, \tau)e^{-s\tau}d\tau \tag{9}$$

Taking the Laplace transform of equation (8) yields

$$\frac{d^2\Theta(\xi, s)}{d\xi^2} = s\Theta(\xi, s) - \theta(\xi, \tau = 0) = s\Theta(s, \xi) - \xi. \tag{10}$$

Equation (10) is an ODE in Θ as a function of ξ, where s is merely a free parameter. It is subject to the following boundary conditions:

$$\Theta(\xi = 0, s) = 0 \tag{11a}$$

and

$$\Theta(\xi = 1, s) = \frac{1}{s} - \frac{1}{\gamma s^2}(1 - e^{-s\gamma}). \tag{11b}$$

Equation (10) may be solved by elementary methods, and its solution is

$$\Theta(\xi, s) = \frac{\xi}{s} - \frac{(1 - e^{-\gamma s})}{\gamma s^2}\frac{\sin h(\xi\sqrt{s})}{\sin h\sqrt{s}}. \tag{12}$$

It can be easily verified that equation (12) satisfies the governing equation, equation (10), and the boundary conditions, equations (11a) and (11b). To obtain the solution of the stated problem, one takes the inverse Laplace transform defined by

$$\theta(\xi, \tau) = L^{-1}(\Theta(\xi, s)) = \frac{1}{2\pi i}\int_{-i\infty+\Gamma}^{i\infty+\Gamma} \Theta(\xi, s)e^{s\tau}ds, \tag{13}$$

where Γ is an arbitrary positive real number. To calculate this integral, the path of the line integral is closed at $|s| \rightarrow \infty$, and the closed contour is evaluated using calculus-of-residues techniques [*Brown and Churchill*, 1996]. The exact solution of equation (5) subject to equations (6)–(8) is

$$\theta(\xi, \tau) = \xi\left(1 - \frac{\tau}{\gamma}\right) + \frac{\xi(1 - \xi^2)}{6\gamma} + \frac{2}{\gamma}\sum_{m=1}^{\infty}\frac{\sin(m\pi\xi)(-1)^m e^{-(m\pi)^2\tau}}{(m\pi)^3},$$
$$0 \le \tau \le \gamma \tag{14}$$

$$\theta(\xi, \tau) = \frac{2}{\gamma}\sum_{m=1}^{\infty}\frac{\sin(m\pi\xi)(-1)^m\left(e^{(-m\pi)^2\tau} - e^{-(m\pi)^2(\tau-\gamma)}\right)}{(m\pi)^3},$$
$$\tau \ge \gamma \ge 0. \tag{15}$$

[10] It is important to verify that equations (14) and (15) provide the same value of θ at $\tau = \gamma$. By setting $\tau = \gamma$ in equation (15), one obtains

$$\begin{aligned}\theta(\xi, \gamma) &= \frac{2}{\gamma}\sum_{m=1}^{\infty}\frac{\sin(m\pi\xi)(-1)^m\left(e^{-(m\pi)^2\gamma} - 1\right)}{(m\pi)^3} \\ &= -\frac{2}{\gamma}\sum_{m=1}^{\infty}\frac{\sin(m\pi\xi)(-1)^m}{(m\pi)^3} \\ &\quad + \frac{2}{\gamma}\sum_{m=1}^{\infty}\frac{\sin(m\pi\xi)(-1)^m e^{-(m\pi)^2\gamma}}{(m\pi)^3}.\end{aligned} \tag{16}$$

The first term of equation (14) is identically zero at $\tau = \gamma$, and it may be shown that the first term in equation (16) is the Fourier series expansion of $\xi(1 - \xi^2)/6\gamma$. Hence equations (14) and (15) are identical for $\tau = \gamma$.

4. Water Flow

[11] The dimensional flow rate (assuming a unit cross-sectional area) in the subsurface is given by

$$q^* = -K\frac{dh}{dx}. \tag{17}$$

Using the dimensionless formulation (equations (4a)–(4c)), one obtains

$$q^* = -K\frac{(h_L - h_0)}{L}\frac{\partial\theta}{\partial\xi}. \tag{18}$$

A dimensionless flow, q, may be defined as

$$q = \frac{q^*}{-K\frac{(h_L - h_0)}{L}} = \frac{\partial\theta}{\partial\xi}. \tag{19}$$

The denominator of the second term in equation (19) is the initial flow rate. From equations (14) and (15), one obtains the dimensionless flow q at a selected time τ and a selected location ξ:
For $\tau \le \gamma$,

$$q(\xi, \tau) = \left(1 - \frac{\tau}{\gamma}\right) + \frac{(1 - 3\xi^2)}{6\gamma} + \frac{2}{\gamma}\sum_{m=1}^{\infty}\frac{(-1)^m\cos(m\pi\xi)e^{-(m\pi)^2\tau}}{(m\pi)^2}, \tag{20}$$

and for $\tau \ge \gamma$,

$$q(\xi, \tau) = \frac{2}{\gamma}\sum_{m=1}^{\infty}\frac{(-1)^m\cos(m\pi\xi)\left(e^{-(m\pi)^2\tau} - e^{-(m\pi)^2(\tau-\gamma)}\right)}{(m\pi)^2}. \tag{21}$$

Note that q (or q^*) is positive in the positive x direction. The flow rate at the groundwater/stream interface is obtained by setting $\xi = 1$ in equations (20) and (21), resulting in

$$q(\xi = 1, \tau) = \left(1 - \frac{\tau}{\gamma}\right) - \frac{1}{3\gamma} + \frac{2}{\gamma}\sum_{m=1}^{\infty}\frac{e^{-(m\pi)^2\tau}}{(m\pi)^2}, \quad \tau \le \gamma. \tag{22}$$

$$q(\xi = 1, \tau) = \frac{2}{\gamma}\sum_{m=1}^{\infty}\frac{e^{-(m\pi)^2\tau} - e^{-(m\pi)^2(\tau-\gamma)}}{(m\pi)^2}, \quad \tau \ge \gamma. \tag{23}$$

The (dimensionless) net outflow volume at the groundwater/stream interface is obtained by integrating the flow q over time, namely,

$$V(\tau) = \int_0^\tau q(\xi = 1, \tau')d\tau', \tag{24}$$

where τ' is an integration variable. In other words, V is the algebraic sum of the water volume that left the aquifer and the water volume that entered the aquifer up to time τ. Inserting equations (22) and (23) in equation (24) results in

$$V(\tau < \gamma) = \tau\left(1 - \frac{\tau}{2\gamma}\right) - \frac{\tau}{3\gamma} + \frac{2}{\gamma}\sum_{m=1}^{\infty}\frac{1 - e^{-(m\pi)^2\tau}}{(m\pi)^4}, \quad 0 \le \tau < \gamma, \tag{25}$$

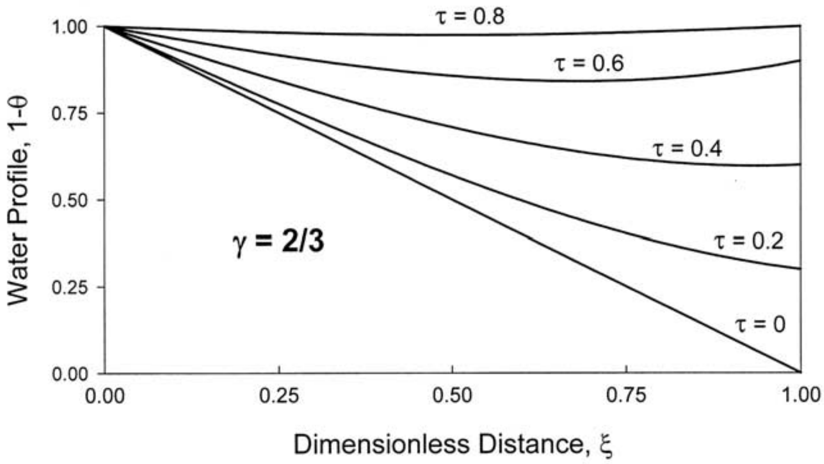

Figure 2. Variation of the surrogate piezometric profile, $1 - \theta$, as a function of the dimensionless space and time (equations (14) and (15)).

$$V_r = V(\tau = \gamma) = \frac{\gamma}{2} - \frac{1}{3} + \frac{2}{\gamma} \sum_{m=1}^{\infty} \frac{1 - e^{-(m\pi)^2\gamma}}{(m\pi)^4}, \quad \tau = \gamma, \quad (26)$$

$$V(\tau > \gamma) = V_r - \frac{2}{\gamma} \sum_{m=1}^{\infty} \frac{\left(1 - e^{-(m\pi)^2\gamma}\right)\left(1 - e^{-(m\pi)^2\,(\tau-\gamma)}\right)}{(m\pi)^4}, \quad \tau > \gamma, \quad (27)$$

where V_r is the net exchange volume at the end of the rise period.

[12] The final net exchange volume (i.e., for $\tau \to \infty$) is obtained from equation (27) (after substituting for V_r from equation (26)) as $V_\infty = \gamma/2 - 1/3$. Thus the critical results are

$$\gamma > 2/3 \Rightarrow V_\infty > 0, \quad (28a)$$

where more water leaves the aquifer than enters it;

$$\gamma < 2/3 \Rightarrow V_\infty < 0, \quad (28b)$$

where less water leaves the aquifer than enters it;

$$\gamma = 2/3 \Rightarrow V_\infty = 0, \quad (28c)$$

where the volume that leaves the aquifer is equal to the volume that enters it. Noting that γ represents the rise period, one may use the relations above to classify the rise as slow rise ($\gamma > 2/3$), fast rise ($\gamma < 2/3$), and neutral rise ($\gamma = 2/3$).

5. Discussion

[13] Figure 2 shows plots of the dimensionless piezometric profile, $(1 - \theta)$, as function of space and time for $\gamma = 2/3$. Figure 3 shows plots of the volume V (equations (26)–(27)) as a function of time for various γ values. At earlier times, groundwater leaves the subsurface (V positive and increasing) until a critical time, $\tau_{\max}$, when V reaches its maximum value, $V_{\max}$, and starts decreasing, which indicates that the outward flow has stopped and stream water is entering the subsurface. The time $\tau_{\max}$ is obtained by setting the time derivative of V, Q ($\xi = 1$, τ), equation (20), to zero. The resulting solution contains the sum over m in the general case. However, because $V_{\max}$ always occurs for $\tau_{\max} < \gamma$, one may neglect the sum over m for $\gamma \geq 1$. This results in

$$\tau_{\max} \approx \gamma - 1/3. \quad (29)$$

The error in estimating $\tau_{\max}$ by equation (29) is less than 2% for $\gamma \geq 1$.

[14] $V_{\max}$ is obtained by setting $\tau = \tau_{\max}$ in equation (25). Using the approximation given by equation (29), one obtains

$$V_{\max} \approx Y/2 - 1/3 + 1/(9\gamma). \quad (30)$$

In such a case the amount of water that enters the subsurface when $\tau \to \infty$ is obtained as

$$V_{\text{in}} = V_\infty - V_{\max} \approx -1/(9\gamma). \quad (31)$$

V_{in} is negative indicating that even at large γ values, a certain volume of stream water will enter the subsurface. Although this volume is small at large γ values (≈ -0.05 at $\gamma = 2$), the chemicals that it typically carries (e.g., dissolved oxygen, nutrients) might be essential for biochemical reactions around the groundwater/stream interface [*Lendvay et al.*, 1998; *Genereux and Slater*, 1999; *Boufadel et al.*, 1999b]. Equation (31) shows that this volume is inversely proportional to γ. The volume that enters the aquifer after the rise period is given by

$$V_{r\infty} = V_\infty - V_r = -\frac{2}{\gamma} \sum_{m=1}^{\infty} \frac{1 - e^{-(m\pi)^2\gamma}}{(m\pi)^4}. \quad (32)$$

[15] Figure 4 shows that the magnitude of $V_{r\infty}$ decreases as γ increases. The maximum value occurs at the intercept (i.e., for $\gamma = 0$) and is obtained by applying l'Hospital rule on equation (32):

$$[V_{r\infty}]_{\gamma=0} = -\frac{2}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \approx -0.3323 \approx -\frac{1}{3}. \quad (33)$$

6. Notation

K hydraulic conductivity.

h piezometric head.

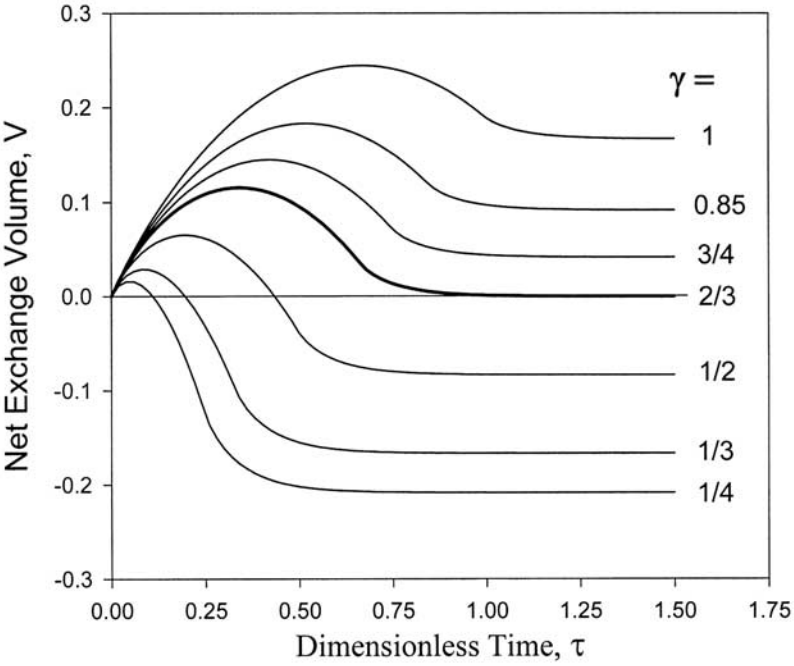

Figure 3. Variation of the net exchange volume at the groundwater/stream interface as a function of τ for various γ values (equations (25)–(27)).

- h_L Minitial piezometric head at right boundary (Figure 1).
- h_0 piezometric head at left boundary (Figure 1).
- L domain length.
- q dimensionless flow rate.
- q^* dimensional flow rate.
- S_0 specific storativity.
- t time.
- t_r rise period, time taken by h (L, t) to reach the value h_0 starting from h_L (Figure 1).
- V dimensionless net exchange volume at the groundwater/stream interface.
- $V_{\max}$ maximum outflow volume.
- V_r dimensionless net exchange volume at the groundwater/stream interface at time t_r.
- $V_{r\infty}$ dimensionless volume that enters the subsurface after the time t_r.
- V_∞ dimensionless net exchange volume at the ground-

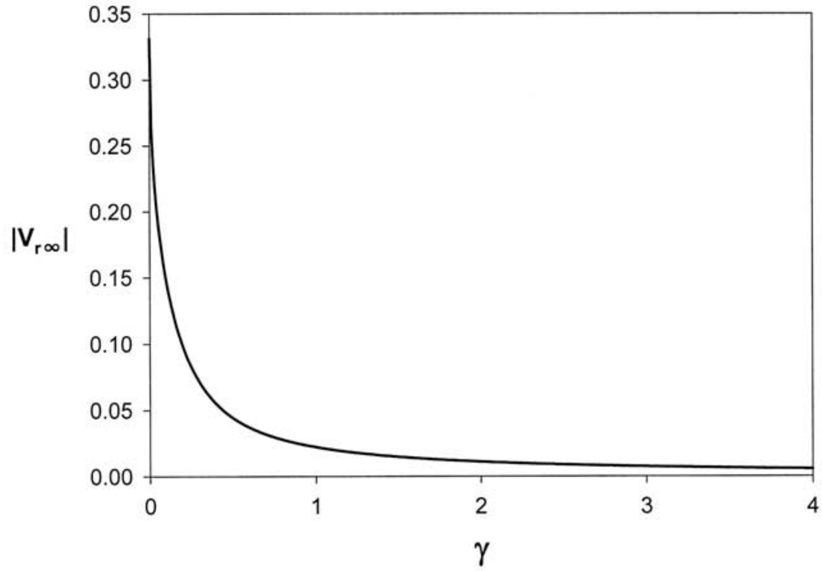

Figure 4. Magnitude of the total volume that enters the aquifer after the rise period as a function of γ (equation (32)).

water/stream interface at an infinite time.

x spatial location.

Greek

α aquifer diffusivity equal to K/S_0.

γ dimensionless rise period equal to $t_r\ \alpha/L^2$.

θ $(h - h_0)/(h_L - h_0)$.

τ dimensionless time equal to $t\ \alpha/L^2$.

τ_{max} dimensionless time at which $V = V_{max}$.

ξ dimensionless spatial location equal to x/L.

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