Two Player Statistical Game with Higher Order Cumulants

Jong-Ha Lee, *Student Member, IEEE*, Chang-Hee Won, *Member, IEEE*,
Ronard W. Diersing, *Member, IEEE*

***Abstract*— In statistical game, we have two objective functions and two players optimize the distribution of the objective function through the cost cumulants. This paper extends the use of second, third, and fourth cumulants into stochastic Nash game. For the case of linear systems with quadratic costs, the equilibrium solutions are determined. In the process, coupled Riccati equations are found. The higher order statistical game is applied to the control of satellite attitude and the results are compared with H_∞ control. The simulations showed that the fourth cumulant case is 26%, 25% and 16% faster than H_∞ control, second cumulant, and third cumulant cases, respectively.**

I. INTRODUCTION

The Linear Quadratic Gaussian (LQG) optimization problem minimizes the mean, which is the first cumulant, of a quadratic cost function [1], [2]. However, the mean or the first cumulant is only one of the cumulants that describe the distribution of a random variable. A more general optimization problem, however, can be formulated to shape the distribution of a cost function. This method of shaping the distribution by optimizing a cost cumulant is called statistical or cost cumulant control. Multi-objective control is a control method in which the control must concern itself with not only one performance index, as in traditional optimal control, but several [3]. The most prevalent multi-objective control is the mixed H_2/H_∞ control, in which the control wishes to minimize an H_2 norm while keeping the H_∞ norm constrained. This approach started in [4], and the Nash game approaches to this problem are proposed in [5]. In [6], two players, a control and a disturbance, are considered. They both wish to minimize their respective performance indices when the other player has played their equilibrium solution. In the stochastic version of this problem, the players then wish to minimize the mean of their cost functions. Previously, the authors investigated *Minimal Cost Variance (MCV)/H_∞* control in [7]. *MCV/H_∞* control method was to combine cost cumulants and *MCV* control to minimize a linear combination of the first two cumulants the mean and the variance while satisfying H_∞ constraint. In this work, *MCV/MCV*, *third/third*, and *fourth/fourth* cumulant control is considered.

Jong-Ha Lee is with the Department of Electrical and Computer Engineering, Temple University, PA 19122, USA jong@temple.edu

Chang-Hee Won is an associate professor of Electrical and Computer Engineering, Temple University, PA 19122, USA cwon@temple.edu

Ronald W. Diersing is an assistant professor of Electrical and Computer Engineering, University of Southern Indiana, IN 47712, USA rwdiersing@usi.edu

To develop more intuition for statistical control, let us consider the second cumulant (variance) case. As pointed out by Mariton in [8], the question of robustness with respect to the underlining stochastic process is important. Also, it is the performance of the sample path that we should be more concerned, and minimal mean does not consider the variance, or the distribution of the cost function, thus it does not guarantee anything about the sample path. However, the cost variance indicates to what extent the performance is spread around its mean value. This variance may play a more important role than the mean in certain applications.

In this paper, second cumulant (variance), third cumulant (skewness), and fourth cumulant (kurtosis) is used to extend stochastic game theory. These cumulant are used to improve the performance of the system. Solutions of higher order statistical game optimization problems are found using Hamilton-Jacobi-Bellman (HJB) equations. We derive necessary (HJB equations) and sufficient (verification theorems) conditions for the high order cumulants. For the case of linear systems with quadratic costs, we found the equilibrium solutions in each culumant case. For the application, the statistical game control through higher order cost cumulant is applied for Low Earth Orbit (LEO) satellite attitude control.

We present the statistical game control problem formulation in Section II. In Section III, Hamilton-Jacobi-Bellman (HJB) equations and Riccati equations for the second, third and fourth cumulant statistical game control are provided. In Section IV, the performance of controls, H_∞ control is compared to that of second, third, and fourth cost cumulant control with real parameters of the Korea Multi Purpose Satellite (KOMPSAT) system. Finally, Section V concludes this paper.

II. PROBLEM FORMULATION

Let $Q_0 = [t_0, t_F) \times \mathbb{R}^n$, $\bar{Q}_0$ denote the closure of Q_0, $T = [t_0, t_F]$, and let $U, V \subset \mathbb{R}^m$ denote a set from which a control applied at any time t is chosen. The system is described as a linear system

$$dx(t) = (A(t)x(t) + B(t)u(t) + D(t)v(t))dt + E(t)d\xi(t), \quad (1)$$

where $t \in [t_0, t_F], x(t_0) = x_0$, and $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in U$ is the control action (player one), $v(t) \in V$ is the disturbance (player two), and $d\xi(t)$ is a Gaussian random process of dimension d with zero mean and covariance of $W(t)dt$. A memoryless feedback *control law* strategy is introduced as $u(t) = \mu(t, x(t)), v(t) = \varphi(t, x(t)),\ t \in T$, where μ and φ are nonrandom functions with random arguments. Now we admit only bounded, Borel measurable feedback strategies, $\mu(t,x) : \bar{Q}_0 \to U$ and $\varphi(t,x) : \bar{Q}_0 \to V$ such that $\mu(t,x)$ and $\varphi(t,x)$ satisfy local Lipschitz conditions and linear growth conditions. A feedback strategy satisfies both of these conditions is called *admissible*. Then a pathwise unique solution process $x(t)$ of (1) exists with probability one, see [2].

This differential equation has two cost functions. The first cost function, J_1, is to be associated with the control u and the second, J_2 is for the disturbance v. The players' cost functions will be assumed to be quadratic;

$$J_1(t_0,x,u,v) = \int_{t_0}^{t_F} (x'(t)Q(t)x(t) + u'(t)R(t)u(t) + v'(t)S(t)v(t))dt \quad (2)$$

$$J_2(t_0,x,u,v) = \int_{t_0}^{t_F} (x'(t)\bar{Q}(t)x(t) + u'(t)\bar{R}(t)u(t) + v'(t)\bar{S}(t)v(t))dt \quad (3)$$

where Q and $\bar{Q}$ are positive semidefinite and R, S, $\bar{R}$, $\bar{S}$ are positive definite. Moments are defined as

$$M_i(t,x,\mu,\varphi) = E\left\{J_1^i(t,x;\mu,\varphi)|x(t) = x\right\}. \quad (4)$$

$$\bar{M}_i(t,x,\mu,\varphi) = E\left\{J_2^i(t,x;\mu,\varphi)|x(t) = x\right\}. \quad (5)$$

From here on, we will use the notation E_{tx} for the conditional expectation. For example, $M_1(t,x;\mu,\varphi) = E_{tx}\{J_1(t,x,u,v)\}$ and $M_2(t,x;\mu,\varphi) = E_{tx}\{J_1^2(t,x,u,v)\}$ are the first and second moments of the cost function $J_1(t,x,u,v)$. Likewise, let $\bar{M}_1(t,x;\mu,\varphi) = E_{tx}\{J_2(t,x,u,v)\}$ and $\bar{M}_2(t,x;\mu,\varphi) = E_{tx}\{J_2^2(t,x,u,v)\}$. The $i-$th cumulants for the J_1 are denoted as $V_i(t,x;\mu,\varphi)$ and $\bar{V}_i(t,x;\mu,\varphi)$ for the J_2.

The two player Nash game is now defined. In Nash (non zero sum) game, there are two performance indices, one for each player, and each player minimizes his index when the other player has played his equilibrium solution. The player cost functions are given in (2) and (3). Here we consider a game where the first player, the control u, minimizes the following performance index,

$$P_1(t,x;\mu,\varphi) = V_i(t,x;\mu,\varphi).$$

The second player, the disturbance v, minimizes the following performance index,

$$P_2(t,x;\mu,\varphi) = \bar{V}_i(t,x;\mu,\varphi).$$

Because both players will be assumed to have full feedback information available to them, $\mathscr{U}_F$ will be the information pattern for the control and $\mathscr{V}_F$ will be the information pattern for the disturbance. Thus, $\mathscr{U}_F$ is the class of all feedback strategies μ already described, and similarly for $\mathscr{V}_F$. The definition of Nash equilibrium solution to the game is given in [3].

Now, we will derive the Hamilton-Jacobi-Bellman (HJB) equation and verification theorems for second, third and fourth cumulant optimization for the first player and J_1. A similar theorems are available for the second player and J_2.

III. Hamilton-Jacobi-Bellman Equation for $V^*(t,x)$

We derive a Hamilton-Jacobi-Bellman (HJB) equation for the second, third, and fourth cumulant statistical game control. The proofs from Theorems 3.1 to 3.6 are given in [3]. First we consider the second cumulant case.

Theorem 3.1: Let $M_1 \in C_p^{1,2}(\bar{Q}_0)$ be an admissible mean cost function, and let M_1 induce a nonempty class $\mathscr{U}_{M_1}$ of admissible control laws. Assume the existence of an optimal control law $\mu = \mu^*_{V_2|M_1}$ and an optimum value function $V_2^* \in C_p^{1,2}(\bar{Q}_0)$. Then the minimal second cumulant (variance) function V_2^* satisfies the following HJB equation.

$$\min_{\mu\in\mathscr{U}_{M_1}} \mathscr{O}(\mu,\varphi^*)[V_2^*(t,x)] + \left\| \frac{\partial V_1(t,x)}{\partial x} \right\|^2_{\sigma W \sigma'} = 0 \quad (6)$$

for $(t,x) \in \bar{Q}_0$, together with the terminal condition, $V_2^*(t_F,x) = 0$.

Theorem 3.2: (Second Cumulant Verification Theorem) Let $M_1 \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be an admissible mean cost function. Let $V_2^* \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be a solution to the partial differential equation

$$\min_{\mu\in\mathscr{U}_{M_1}} \mathscr{O}(\mu,\varphi^*)[V_2^*(t,x)] + + \left\| \frac{\partial V_1(t,x)}{\partial x} \right\|^2_{\sigma W \sigma'} = 0, \quad (7)$$

where $V_2^*(t_f,x) = 0$. Then $V_2^*(t,x) \leq Var\{J_1(t,x,\mu(t,x))\}$ for all $\mu \in \mathscr{U}_{M_1}$ and $(t,x) \in Q$. If in addition, such $\mu^*_{V_2|M_1}$ satisfies the following equation,

$$\mathscr{O}(\mu,\varphi^*)[V_2^*(t,x)] = \min_{\mu\in\mathscr{U}_{M_1}} \{\mathscr{O}(\mu,\varphi^*)[V_2^*(t,x)]\},$$

then $V_2^*(t,x) = Var\{J_1(t,x,\mu^*_{V_2|M_1}(t,x),\varphi^*)\}$ and $\mu^*_{V_2|M_1}$ is the control's equilibrium solution for the two cumulant, two player game.

We present the third cumulant HJB equation. The following theorem is a necessary condition for the optimal controller, which is obtained via a HJB equation. If an optimal controller exists then it will necessarily satisfy the following theorem.

Theorem 3.3: Let $M_1 \in C_p^{1,2}(\bar{Q}_0)$ be an admissible mean cost function, and let M_1 induce a nonempty class $\mathscr{U}_{M_1}$ of admissible control laws. Assume the existence of an optimal control law $\mu = \mu^*_{V_3|M_1}$ and an optimum value function $V_3^* \in C_p^{1,2}(\bar{Q}_0)$. Then the minimal third cost cumulant (skewness) function V_3^* satisfies the following HJB equation.

$$\min_{\mu\in\mathscr{U}_{M_1}} \left\{ \begin{array}{l} \mathscr{O}(\mu,\varphi^*)[V_3^*(t,x)] \\ +3tr\left(\sigma W \sigma' \left(\frac{\partial V_1(t,x)}{\partial x}\right)\left(\frac{\partial V_2(t,x)}{\partial x}\right)'\right) \end{array} \right\} = 0 \quad (8)$$

for $(t,x) \in \bar{Q}_0$, together with the terminal condition, $V_3^*(t_F,x) = 0$.

The following theorem is a sufficient condition for the optimal controller.

Theorem 3.4: (Third Cumulant Verification Theorem) Let $M_1 \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be an admissible mean cost function.

Let $V_3^* \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be a solution to the partial differential equation

$$\begin{aligned} &\min_{\mu \in \mathscr{U}_{M_1}} \mathscr{O}(\mu, \varphi^*)[V_3^*(t,x)] \\ &+3tr\left(\sigma W \sigma' \left(\frac{\partial V_1(t,x)}{\partial x}\right)\left(\frac{\partial V_2(t,x)}{\partial x}\right)'\right) = 0. \end{aligned} \tag{9}$$

Then $V_3^*(t,x)$ is less than or equal to the third cumulant of the cost $J(t,x;\mu(t,x),\varphi^*)$ for all $\mu \in \mathscr{U}_{M_1}$ and $(t,x) \in Q$.

Here, we present the fourth cumulant (kurtosis) HJB equation as a necessary condition for optimality.

Theorem 3.5: Let $M_1 \in C_p^{1,2}(\bar{Q}_0)$ be an admissible mean cost function, and let M_1 induce a nonempty class $\mathscr{U}_{M_1}$ of admissible control laws. Assume the existence of an optimal control law $\mu = \mu^*_{V_4|M_1}$ and an optimum value function $V_4^* \in C_p^{1,2}(\bar{Q}_0)$. Then the minimal fourth cost cumulant (kurtosis) function V_4^* satisfies the following HJB equation,

$$\begin{aligned} &\min_{\mu \in \mathscr{U}_{M_1}} \mathscr{O}(\mu, \varphi^*)[V_4^*(t,x)] + 3\left\|\frac{\partial V_2}{\partial x}\right\|^2_{\sigma W \sigma'} \\ &+4tr\left(\sigma W \sigma' \left(\frac{\partial V_1(t,x)}{\partial x}\right)\left(\frac{\partial V_3(t,x)}{\partial x}\right)'\right) = 0, \end{aligned} \tag{10}$$

for $(t,x) \in \bar{Q}_0$, together with the terminal condition, $V_4^*(t_F,x) = 0$.

The following sufficient condition for optimality is presented in the form of a verification theorem.

Theorem 3.6: (Fourth Cumulant Verification Theorem)

Let $M_1 \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be an admissible mean cost function. Let $V_4^* \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be a solution to the partial differential equation

$$\begin{aligned} &\min_{\mu \in \mathscr{U}_{M_1}} \mathscr{O}(\mu, \varphi^*)\,[V_4^*(t,x)] + \left\|\frac{\partial V_2(t,x)}{\partial x}\right\|^2_{\sigma W \sigma'} \\ &+3tr\left(\sigma W \sigma' \left(\frac{\partial V_1(t,x)}{\partial x}\right)\left(\frac{\partial V_2(t,x)}{\partial x}\right)'\right) = 0. \end{aligned} \tag{11}$$

Then $V_4^*(t,x)$ is less than or equal to the fourth cumulant of the cost $J_1(t,x;\mu(t,x),\varphi^*)$ for all $\mu \in \mathscr{U}_{M_1}$ and $(t,x) \in Q$.

We are solving Nash (non zero sum) game, where two players have different cost functions. So we need Nash equilibrium. From now we determine second, third and fourth cumulant Nash equilibrium solutions. First we consider the second cumulant case. We assume that the value functions are quadratic. That is $V_1(t,x) = x'\mathscr{V}_1(t)x + v_1(t)$, and similarly with $V_2(t,x)$, $\bar{V}_1(t,x)$, $\bar{V}_2(t,x)$ where $\mathscr{V}_1$, $\mathscr{V}_2$, $\bar{\mathscr{V}}_1$, $\bar{\mathscr{V}}_2$ are matrix functions of time and m_1, m_2, $\bar{m}_2$, $\bar{m}_2$ are scalar functions of time. Recall the second cumulant HJB equation. For the control, we have

$$\begin{aligned} \min_{\mu \in \mathscr{U}_F} \{&\gamma_1[x'\dot{V}_1x + \dot{v}_1 + 2(Ax + B\mu + D\varphi^*)'V_1x \\ &+ x'Qx + \mu'R\mu + \varphi^{*'}S\varphi^*] + \gamma_2[x'\dot{V}_2x + \dot{v}_2 \\ &+ 2(Ax + B\mu + D\varphi^*)'V_2x + 4V_1EWE'V_1] \\ &+ tr(EWE'(\gamma_1V_1 + \gamma_2V_2)\} = 0 \end{aligned} \tag{12}$$

and minimizing this gives

$$u(t)^* = \mu^*(t,x(t)) = -R^{-1}B'(t)[\mathscr{V}_1(t) + \frac{\gamma_2}{\gamma_1}\mathscr{V}_2(t)]x(t) \tag{13}$$

which is the form of the second cumulant controller's Nash equilibrium solution. Similarly, for the disturbance

$$\begin{aligned} \min_{\varphi \in \mathscr{V}_F} \{&\bar{\gamma}_1[x'\dot{\bar{V}}_1x + \dot{\bar{v}}_1 + 2(Ax + B\mu^* + D\varphi)'\bar{V}_1x \\ &+ x'\bar{Q}x + \mu^{*'}\bar{R}\mu^* + \varphi'\bar{S}\varphi] + \bar{\gamma}_2[x'\dot{\bar{V}}_2x + \bar{v}_2 \\ &+ 2(Ax + B\mu^* + D\varphi)'\bar{V}_2x + 4\bar{V}_1EWE'\bar{V}_1] \\ &+ tr(EWE'(\bar{\gamma}_1\bar{V}_1 + \bar{\gamma}_2\bar{V}_2)\} = 0 \end{aligned} \tag{14}$$

which by minimization yields

$$v(t)^* = \varphi^*(t,x(t)) = -\bar{S}^{-1}D'(t)[\bar{\mathscr{V}}_1(t) + \frac{\bar{\gamma}_2}{\bar{\gamma}_1}\bar{\mathscr{V}}_2(t)]x(t) \tag{15}$$

Using this Nash equilibrium solution (μ^*,φ^*), we can determine the Riccati equations by substitution. Consider the mean of the control's cost function

$$\begin{aligned} &\dot{\mathscr{V}}_1 + A'\mathscr{V}_1 + \mathscr{V}_1A + Q - [\bar{\gamma}_1\bar{\mathscr{V}}_1 + \bar{\gamma}_2\bar{\mathscr{V}}_2]D\bar{S}^{-1}D'\mathscr{V}_1 \\ &\quad - \mathscr{V}_1D\bar{S}^{-1}D'[\bar{\gamma}_1\bar{\mathscr{V}}_1 + \bar{\gamma}_2\bar{\mathscr{V}}_2] - [\gamma_1\mathscr{V}_1 + \gamma_2\mathscr{V}_1]BR^{-1}B'\mathscr{V}_1 \\ &\quad - \mathscr{V}_1BR^{-1}B'[\gamma_1\mathscr{V}_1 + \gamma_2\mathscr{V}_2] + [\gamma_1\mathscr{V}_1 + \gamma_2\mathscr{V}_1]BR^{-1}B' \\ &\quad [\gamma_1\mathscr{V}_1 + \gamma_2\mathscr{V}_1] + [\bar{\gamma}_1\bar{\mathscr{V}}_1 + \bar{\gamma}_2\bar{\mathscr{V}}_2]D\bar{S}^{-1}S\bar{S}^{-1}D' \\ &\quad [\bar{\gamma}_1\bar{\mathscr{V}}_1 + \bar{\gamma}_2\bar{\mathscr{V}}_2] = 0 \end{aligned} \tag{16}$$

where $\mathscr{V}_1(t_f) = Q_f$. Next we derive an expression for the variance of the control's cost function.

$$\begin{aligned} &\dot{\mathscr{V}}_2 + A'\mathscr{V}_2 + \mathscr{V}_2A - [\bar{\gamma}_1\bar{\mathscr{V}}_1 + \bar{\gamma}_2\bar{\mathscr{V}}_2]D\bar{S}^{-1}D'\mathscr{V}_1 \\ &\quad - \mathscr{V}_2D\bar{S}^{-1}D'[\bar{\gamma}_1\bar{\mathscr{V}}_1 + \bar{\gamma}_2\bar{\mathscr{V}}_2] - [\gamma_1\mathscr{V}_1 + \gamma_2\mathscr{V}_2]BR^{-1}B'\mathscr{V}_2 \\ &\quad - \mathscr{V}_2BR^{-1}B'[\gamma_1\mathscr{V}_1 + \gamma_2\mathscr{V}_2] + 4\mathscr{V}_1EWE'\mathscr{V}_1 = 0 \end{aligned} \tag{17}$$

with $\mathscr{V}_2(t_f) = 0$.

Expressions for the mean and the variance of the disturbance's cost function are given by

$$\begin{aligned} &\dot{\bar{\mathscr{V}}}_1 + A'\bar{\mathscr{V}}_1 + \bar{\mathscr{V}}_1A + \bar{Q} - [\bar{\gamma}_1\bar{\mathscr{V}}_1 + \bar{\gamma}_2\bar{\mathscr{V}}_2]D\bar{S}^{-1}D'\bar{\mathscr{V}}_1 \\ &\quad - \bar{\mathscr{V}}_1D\bar{S}^{-1}D'[\bar{\gamma}_1\bar{\mathscr{V}}_1 + \bar{\gamma}_2\bar{\mathscr{V}}_2] - [\gamma_1\mathscr{V}_1 + \gamma_2\mathscr{V}_1]BR^{-1}B'\bar{\mathscr{V}}_1 \\ &\quad - \bar{\mathscr{V}}_1BR^{-1}B'[\gamma_1\mathscr{V}_1 + \gamma_2\mathscr{V}_2] + [\gamma_1\mathscr{V}_1 + \gamma_2\mathscr{V}_1]BR^{-1}\bar{R}R^{-1}B' \\ &\quad [\gamma_1\mathscr{V}_1 + \gamma_2\mathscr{V}_1] + [\bar{\gamma}_1\bar{\mathscr{V}}_1 + \bar{\gamma}_2\bar{\mathscr{V}}_2]D\bar{S}^{-1}D'[\bar{\gamma}_1\bar{\mathscr{V}}_1 + \bar{\gamma}_2\bar{\mathscr{V}}_2] = 0 \end{aligned} \tag{18}$$

where $\bar{\mathscr{V}}_1(t_f) = \bar{Q}_f$ and

$$\begin{aligned} &\dot{\bar{\mathscr{V}}}_2 + A'\bar{\mathscr{V}}_2 + \bar{\mathscr{V}}_2A - [\bar{\gamma}_1\bar{\mathscr{V}}_1 + \bar{\gamma}_2\bar{\mathscr{V}}_2]D\bar{S}^{-1}D'\bar{\mathscr{V}}_1 \\ &\quad - \bar{\mathscr{V}}_2D\bar{S}^{-1}D'[\bar{\gamma}_1\bar{\mathscr{V}}_1 + \bar{\gamma}_2\bar{\mathscr{V}}_2] - [\gamma_1\mathscr{V}_1 + \gamma_2\mathscr{V}_2]BR^{-1}B'\bar{\mathscr{V}}_2 \\ &\quad - \bar{\mathscr{V}}_2BR^{-1}B'[\gamma_1\mathscr{V}_1 + \gamma_2\mathscr{V}_2] + 4\bar{\mathscr{V}}_1EWE'\bar{\mathscr{V}}_1 = 0 \end{aligned} \tag{19}$$

with $\bar{\mathscr{V}}_2(t_f) = 0$. If these Riccati equations are satisfied, then we know the optimal strategy (μ^*,φ^*) are given in (13) and (15). This leads to the following theorem.

Next we consider the third cumulant case of the Nash equilibrium solutions. We assume the quadratic value functions $V_3(t,x) = x'\mathscr{V}_3(t)x + v_3(t)$ and $\bar{V}_3(t,x) = x'\bar{\mathscr{V}}_3(t)x + \bar{v}_3(t)$.

Recall the third cumulant HJB equation for the control, it gives

$$\begin{aligned}\min_{\mu\in\mathscr{U}_F}\{&\gamma_1[x'\dot{V}_1x+\dot{v}_1+2(Ax+B\mu+D\varphi^*)'V_1x\\&+x'Qx+\mu'R\mu+\varphi^{*'}S\varphi^*]+\gamma_3[x'\dot{V}_3x+\dot{v}_3\\&+2(Ax+B\mu+D\varphi^*)'V_3x+12V_2EWE'V_1]\\&+tr(EWE'(\gamma_1V_1+\gamma_3V_3)\}=0\end{aligned} \tag{20}$$

and minimizing gives the control's Nash equilibrium solution.

$$u(t)^*=\mu^*(t,x(t))=-R^{-1}B'(t)[\mathscr{V}_1(t)+\frac{\gamma_3}{\gamma_1}\mathscr{V}_3(t)]x(t) \tag{21}$$

The third cumulant HJB equation for the disturbance is

$$\begin{aligned}\min_{\varphi\in\mathscr{V}_F}\{&\bar{\gamma}_1[x'\dot{\bar{V}}_1x+\dot{\bar{v}}_1+2(Ax+B\mu^*+D\varphi)'\bar{V}_1x\\&+x'\bar{Q}x+\mu^{*'}\bar{R}\mu^*+\varphi'\bar{S}\varphi]+\bar{\gamma}_3[x'\dot{\bar{V}}_3x+\dot{\bar{v}}_3\\&+2(Ax+B\mu^*+D\varphi)'\bar{V}_3x+12\bar{V}_2EWE'\bar{V}_1]\\&+tr(EWE'(\bar{\gamma}_1\bar{V}_1+\bar{\gamma}_3\bar{V}_3)\}=0\end{aligned} \tag{22}$$

and minimization this gives

$$v(t)^*=\varphi^*(t,x(t))=-\bar{S}^{-1}D'(t)[\bar{\mathscr{V}}_1(t)+\frac{\bar{\gamma}_3}{\bar{\gamma}_1}\bar{\mathscr{V}}_3(t)]x(t) \tag{23}$$

We derive an expression for the skewness of the control's cost function.

$$\begin{aligned}&\dot{\mathscr{V}}_3+A'\mathscr{V}_3+\mathscr{V}_3A-[\bar{\gamma}_1\bar{\mathscr{V}}_1+\bar{\gamma}_3\bar{\mathscr{V}}_3]D\bar{S}^{-1}D'\mathscr{V}_1\\&-\mathscr{V}_3D\bar{S}^{-1}D'[\bar{\gamma}_1\bar{\mathscr{V}}_1+\bar{\gamma}_3\bar{\mathscr{V}}_3]-[\gamma_1\mathscr{V}_1+\gamma_3\mathscr{V}_3]BR^{-1}B'\mathscr{V}_3\\&-\mathscr{V}_3BR^{-1}B'[\gamma_1\mathscr{V}_1+\gamma_3\mathscr{V}_3]+12\mathscr{V}_2EWE'\mathscr{V}_1=0\end{aligned} \tag{24}$$

with $\mathscr{V}_3(t_f)=\bar{Q}_f$. Similarly, the expression for the skewness of the disturbance's cost function is given by follows.

$$\begin{aligned}&\dot{\bar{\mathscr{V}}}_3+A'\bar{\mathscr{V}}_3+\bar{\mathscr{V}}_3A-[\bar{\gamma}_1\bar{\mathscr{V}}_1+\bar{\gamma}_3\bar{\mathscr{V}}_3]D\bar{S}^{-1}D'\bar{\mathscr{V}}_1\\&-\bar{\mathscr{V}}_3D\bar{S}^{-1}D'[\bar{\gamma}_1\bar{\mathscr{V}}_1+\bar{\gamma}_3\bar{\mathscr{V}}_3]-[\gamma_1\mathscr{V}_1+\gamma_3\mathscr{V}_3]BR^{-1}B'\bar{\mathscr{V}}_3\\&-\bar{\mathscr{V}}_3BR^{-1}B'[\gamma_1\mathscr{V}_1+\gamma_3\mathscr{V}_3]+12\bar{\mathscr{V}}_2EWE'\bar{\mathscr{V}}_1=0\end{aligned} \tag{25}$$

The fourth cumulant case is similar as the second and third cumulant case. The controller's Nash equilibrium solution as

$$u(t)^*=\mu^*(t,x(t))=-R^{-1}B'(t)[\mathscr{V}_1(t)+\frac{\gamma_4}{\gamma_1}\mathscr{V}_4(t)]x(t) \tag{26}$$

Similarly, for the disturbance

$$v(t)^*=\varphi^*(t,x(t))=-\bar{S}^{-1}D'(t)[\bar{\mathscr{V}}_1(t)+\frac{\bar{\gamma}_4}{\bar{\gamma}_1}\bar{\mathscr{V}}_4(t)]x(t) \tag{27}$$

IV. Satellite Attitude Control Application

In this section, we investigate multi-objective statistical game control performance in the satellite attitude control. We consider Korea Multi Purpose Satellite (KOMPSAT) system with four reaction wheels and three thrusters for the application of the developed method [13]. The general nonlinear satellite attitude dynamic model is given by

$$\begin{aligned}I_g\underline{\dot{\omega}}=&-\underline{\omega}\times(I_t\underline{\omega}+L'I_w\underline{\Omega})-L'\underline{\tau}_{wheel}\\&+\underline{\tau}_{thruster}+\underline{v}\end{aligned} \tag{28}$$

The control $\underline{u}$ is given by the torque due to reaction wheels and thrusters, $\underline{u}=[\underline{\tau}_{wheel},\underline{\tau}_{thruster}]'$. The disturbance $\underline{v}$ is due to the magnetic field, solar radiation pressure, and atmospheric drag. The states are given as $\underline{x}=[\phi,\theta,\psi,\omega_x,\delta\omega_y,\omega_z,\Omega_1,\Omega_2,\Omega_3,\Omega_4]'\equiv[\underline{\phi_c},\underline{\omega},\underline{\Omega}]'$ where $\omega_x,\omega_y,\omega_z$ are the angular velocities in Body Fixed Coordinate (BFC) system, Ω_i are the wheel speeds, ϕ is the roll Euler angle, θ is the pitch Euler angle, and ψ is the yaw Euler angle. In addition, the following are defined:

1) n: Orbital rate
2) I_t: Total amount of inertia for satellite body (3 × 3)
3) I_w: Wheel moment of inertia matrix (4 × 4)
4) $I_g=I_t-L'I_wL$: Total moment inertial minus moment of inertia of the wheels (3 × 3)
5) L: Wheel orientation matrix (4 × 3)

$$L=\begin{bmatrix}\cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta\\ -\sin\alpha\sin\beta & \cos\alpha\sin\beta & \cos\beta\\ -\cos\alpha\sin\beta & -\sin\alpha\sin\beta & \cos\beta\\ \sin\alpha\sin\beta & -\cos\alpha\sin\beta & \cos\beta\end{bmatrix} \tag{29}$$

Here α and β are the angles of the reaction wheel. α is 45 deg. and β is 54.74 deg. To have zero initial conditions we let $\delta\omega_y=\omega_y+n$, and find the general linear equation form of (28) for any Torque Equilibrium Attitude (TEA) of the states. For the simplicity sake, we assume the case of the attitude hold mode where the TEA values are fixed at the following values: $\underline{x}_0=[0,0,0,0,0,0,0,0,0,0]'$. Assuming the gravity gradient torque for a point mass, we have the following linearized equations.

$$\begin{aligned}\underline{\dot{\omega}}\equiv&I_g^{-1}nN_1\underline{\omega}+I_g^{-1}nN_2\underline{\Omega}+I_g^{-1}3n^2N_3\underline{\phi_c}\\&-I_g^{-1}L'\underline{\tau}_{wheel}+I_g^{-1}\underline{\tau}_{thruster}+I_g^{-1}\underline{v},\end{aligned} \tag{30}$$

$$\underline{\dot{\Omega}}=I_w^{-1}\underline{\tau}_{wheel}-L\underline{\dot{\omega}}, \tag{31}$$

and

$$\begin{aligned}\underline{\dot{\phi_c}}=&[0,0,-n]'\phi+[n,0,0]'\psi+[1,0,0]'\omega_x\\&+[0,1,0]'(\omega_y+n)+[0,0,1]'\omega_z.\end{aligned} \tag{32}$$

Because $\delta\omega_y=\omega_y+n$, we have

$$\begin{aligned}\dot{\phi}&=n\psi+\omega_x\\ \dot{\theta}&=\omega_y+n=\delta\omega_y\\ \dot{\psi}&=-n\phi+\omega_z\end{aligned} \tag{33}$$

Using (30), (31), and (33), we can obtain a linearized differential equation of (1). The detailed description of the model is given in [7]

In the following simulations, each cumulant case is minimized while the mean is kept at a pre-specified level. Figure 1 shows the angular velocity ω_x versus time graph for the H_∞ control, second cumulant, third cumulant, and fourth cumulant cases. Notice that if $\gamma_{\underline{1}}$ is $\underline{1}$, the Riccati equations have unique solutions $\mathscr{V}_1$, $\mathscr{V}_2$, $\bar{\mathscr{V}}_1$, $\bar{\mathscr{V}}_2$ when γ_2 is less than 1.0*e*-6 in the second cumulant case, γ_3 is less than 1.0*e*-11 in the third cumulant case, and γ_4 is less than 1.0*e*-17 in the fourth cumulant case. In the current study, we set γ_2=1.0*e*-7 for the second cumulant case, γ_3=1.0*e*-12 for the third cumulant case, and γ_4=1.0e-18 for the fourth cumulant case. In fact the value of γ_2, γ_3 and γ_4 can be any number at least the Riccati equations can be solvable. The full relationship between γ_2, γ_3, γ_4 values and the control method performance will be remaining for the future work.

We define our settling time of angular velocity ω_x to be within $\pm$2.0e-4 deg/sec. Table I shows the angular velocity ω_x and the settling time for four different control methods. It shows that the settling time in the fourth cumulant case is 29.93 sec and 16.63 sec faster than that of the second cumulant and third cumulant case, respectively.

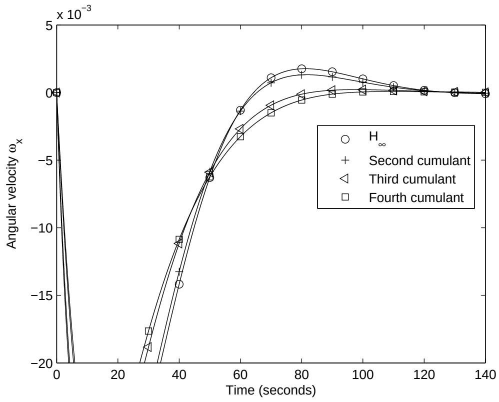

Fig. 1. Angular velocity ω_x versus time

TABLE I

ANGULAR VELOCITY ω_x VERSUS SETTLING TIME FOR FOUR DIFFERENT CONTROL METHODS

Control Method	$\omega_x(deg/sec)$	Time (sec)
H_∞	2.0e-4	118.86
Second cumulant case	2.0e-4	117.19
Third cumulant case	2.0e-4	103.89
Fourth cumulant case	2.0e-4	87.26

Figure 2 shows the roll Euler angle ϕ versus time graph. In this plot, we notice that there exists some undershoot for the H_∞ control and second cumulant case but not for the third cumulant and fourth cumulant case. The Euler angles due to the third cumulant and fourth cumulant cases decrease monotonically toward zero. Table II shows the times when the roll angle reaches within settling value 1.0e-3 deg. Once again, the H_∞ control gives the slowest time and fourth cumulant case gives the fastest time.

TABLE II

ROLL ANGLE ϕ VERSUS TIME OF FOUR DIFFERENT CONTROL METHODS

Control Method	$\phi(deg)$	Time (sec)
H_∞	1.0e-3	156.25
Second cumulant case	1.0e-3	118.81
Third cumulant case	1.0e-3	118.59
Fourth cumulant case	1.0e-3	115.70

The state wheel speed Ω_1 is also simulated in Figure 3. In this plot, one notice that fourth cumulant case has the

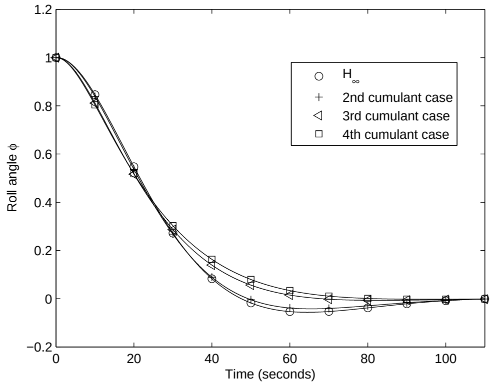

Fig. 2. Roll angle ϕ versus time

smallest undershoot at -2.39e-8, and has the fastest settling time. H_∞ control has the largest undershoot at -1.04e-5, and has the slowest settling time. The undershoot and the settling time of the second cumulant case and third cumulant case is in between H_∞ control and fourth cumulant case. We note that the other three wheel speeds Ω_2, Ω_3, and Ω_4 achieve the similar performance as Ω_1.

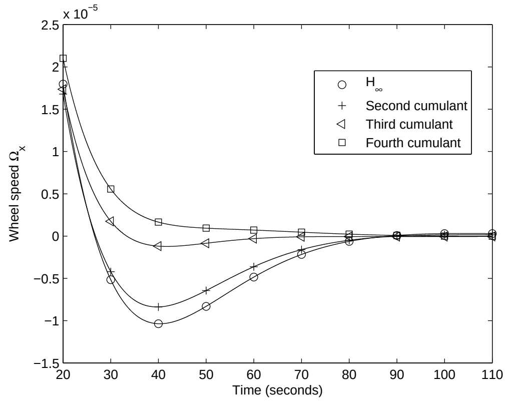

Fig. 3. Wheel speed Ω_1 versus time

TABLE III

REACTION WHEEL SPEED Ω_1 AND TIME OF THREE CONTROL METHODS

Control Method	Ω_1	Time (sec)
H_∞	2.0e-8	147.83
Second cumulant case	2.0e-8	145.44
Third cumulant case	2.0e-8	123.89
Fourth cumulant case	2.0e-8	96.48

We have two control actions given by four reaction wheel torques $\underline{\tau}_{wheel}$ and three thruster torques $\underline{\tau}_{thruster}$. Figure 4 plots the magnitude of $\underline{\tau}_{wheel}$ versus time and Figure 5 plots the magnitude of $\underline{\tau}_{thrusters}$ versus time. Table IV shows the amount of $\underline{\tau}_{wheel}$ and $\underline{\tau}_{thrusters}$ for the specific times. We choose the times to match the settling time of Table I. At those times, we found that the magnitude of $\underline{\tau}_{wheel}$ and

$\tau_{thrusters}$ value of fourth cumulant control is the smallest and H_∞ control is the largest. Second cumulant and third cumulant cases are in between the two methods.

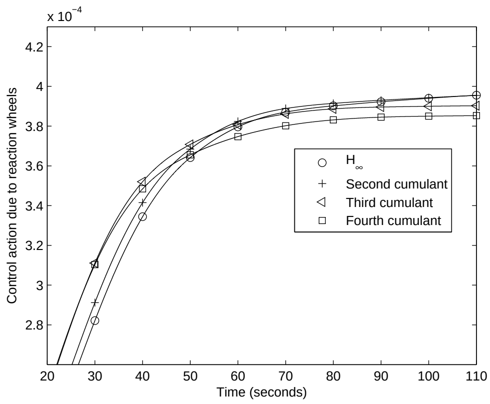

Fig. 4. Control action due to reaction wheels versus settling time

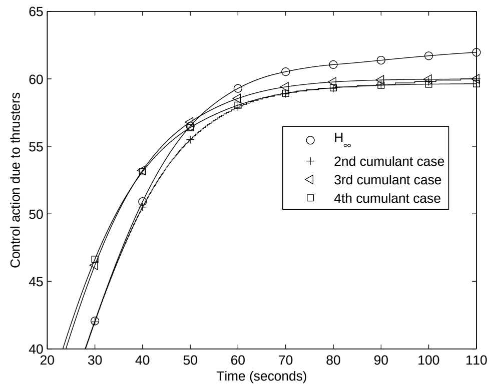

Fig. 5. Control action due to thrusters versus time

TABLE IV
CONTROL ACTION DUE TO REACTION WHEELS AND TRUSTERS VERSUS TIME

Control Method	Time	$\lvert\tau_{wheel}\rvert$	$\lvert\tau_{thruster}\rvert$
H_∞	118.86	3.97e-4	6.21e1
Second cumulant case	117.19	3.96e-4	6.21e1
Third cumulant case	103.89	3.90e-4	6.00e1
Fourth cumulant case	87.26	3.84e-4	5.95e1

We provide the stability results. Table V shows the maximum real part of the closed loop system poles for all four control methods. Compare to the H_∞ control, statistical game control has the smaller closed loop eivenvalues and higher stability margin. Among the statistical game control, third and fourth cumulant cases have the higher stability margin compare to the second cumulant case.

Finally, we provide the computational time of each algorithm. In order to obtain the computational time, we ran satellite attitude control simulation ten times and obtained the average computational time. Table VI shows the required mean simulation time and its standard deviation of all four control methods. From the result, we notice that the fourth cumulant case requires the largest computational time and H_∞ control requires the smallest time.

TABLE V
ROBUST STABILITY COMPARISON

Control Method	Max. Real Part
H_∞	-4.84e-2
Second cumulant case	-4.90e-2
Third cumulant case	-5.35e-2
Fourth cumulant case	-5.34e-2

TABLE VI
COMPUTATIONAL TIME COMPARISON

Control Method	Mean computational time	Std.
H_∞	1.77e-1	3.2e-2
Second cumulant case	2.05e-1	2.4e-2
Third cumulant case	2.12e-1	1.8e-2
Fourth cumulant case	2.49e-1	2.1e-2

V. CONCLUSIONS

In this paper, multi-objective statistical game control method for satellite attitude control is studied. The performance comparison of second, third and fourth cumulant cases and classical control method H_∞ control is provided. We achieved the performance improvement with less control action by using higher order statistical game control.

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