Output Feedback Multiobjective Cumulant Control with Structural Applications

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***Abstract*— Recently, a cumulant generalization of H_2/H_∞ control has been given for the state feedback problem. This paper extends those results to the output feedback case. The Nash game approach to the H_2/H_∞ problem is used. Sufficient conditions for two problems are determined. In the first problem the one player, the control, has only partial state information, while the other player, the disturbance, has full state information. In the other problem both players only have information based upon estimates of the state. Coupled Riccati equations for both cases are given, along with equilibrium solutions. The results are also applied to the first generation structural benchmark for buildings under seismic excitation.**

I. Introduction

Cumulants have been used in control with encouraging results [9], [10], [11], [13]. Their use has been particularly interesting for structural vibration problems. Furthermore, cumulants have also been used in game theory [3], [9]. In [3], H_2/H_∞ was generalized by the use of cumulants and the Nash game. These problems were developed for the case in which both players had full state feedback information. However, the full information of the state is not always available for feedback. Actually, quite often, only a set of output signals, which do not give full information about the states, is available.

Here, we assume that the players do not have full state feedback information. We will first give a definition of the problem. In previous work on full state feedback, [3], the problem developed from a nonlinear system with non-quadratic costs. Here, however, the linear system, quadratic cost assumption will be taken from the beginning. With the system defined, the first problem that will be discussed is one in which the control wishes to minimize a linear combination of k cumulants of its cost function, based upon state estimates, while the disturbance has full state information available for its decision. Then, the case in which both players only have information from state estimates will be examined. Finally, output feedback 3-cumulant multi-objective control will be applied to the first generation structural benchmark.

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II. Problem Definition

In the previous work [3], the problem began in a nonlinear framework with non-quadratic costs. However, in the output feedback case, the linear system and quadratic cost case will be considered from the onset. The linear system in question is given by

$$\begin{aligned} dx(t) = &(A(t)x(t) + B(t)u(t))\,dt \\ &+ D(t)w(t)dt + E(t)d\xi(t) \end{aligned} \tag{1}$$

with $x(t_0) = x_0$ being known. The matrices A, B, D, and E are $n\times n$, $n\times m$, $n\times p$, and $n\times q$ respectively, with continuous entries on the interval $[t_0, t_f]$. The process noise ξ is a Wiener process on $(\Omega, \mathcal{F}, P)$. Also, ξ has an autocorrelation function of

$$E\{[\xi(t) - \xi(\tau)][\xi(t) - \xi(\tau)]'\} = W|t - \tau|.$$

The control will have only measurements available, not the full state information that is available to the other player, the disturbance w. The measurements y are given by

$$dy(t) = C(t)x(t)dt + dv(t) \tag{2}$$

where C is a $n \times r$ matrix with continuous entries on the interval $[t_0, t_f]$ and v is a Wiener process with autocorrelation

$$E\{[v(t) - v(\tau)][v(t) - v(\tau)]'\} = \Xi|t - \tau|.$$

The crosscorrelation between ξ and v is given as

$$E\{[\xi(t) - \xi(\tau)][v(t) - v(\tau)]'\} = \Gamma|t - \tau|,$$

where $W - \Gamma\Xi^{-1}\Gamma' > 0$.

The cost functions are

$$\begin{aligned} J_1(t_0, x_0; u, w) &= \int_{t_0}^{t_f} ||z_1(t)||^2 dt \\ J_2(t_0, x_0; u, w) &= \int_{t_0}^{t_f} \left(\delta^2||w(t)||^2 - ||z_2(t)||^2\right) dt \end{aligned} \tag{3}$$

where δ is a positive constant and the minimization of the mean of J_2 by w corresponds with constraining the H_∞ norm, $||T_{z_2w}||_\infty \leq \delta$, see [2], [8], also for the state feedback cumulant case see [3]. Also, z_1, z_2 are regulated outputs given by

$$\begin{aligned} z_1(t) =& C_1(t)x(t) + D_1(t)u(t) \\ z_2(t) =& C_2(t)x(t) + D_2(t)u(t). \end{aligned} \tag{4}$$

Furthermore for $i = 1, 2$, the matrices C_i, D_i satisfy $C_i'C_i = Q_i, C_i'D_i = 0, D_i'D_i = R_i$. It will be assumed that R_1 is a positive definite matrix.

III. Output Feedback MCC

In this section we develop an output feedback version of the multi-objective cumulant control problem. The disturbance will have full state information available, while the control will only have partial information of the state. An assumption made in this work is that the control strategies are linear. In fact, the feedback control strategies are assumed to be of the form

$$u(t) = \mu(t, \hat{x}(t)) = K(t)\hat{x}(t) \tag{5}$$

where the control is dependent on an estimate of the state x, $\hat{x}(t) = E\{x(t)|\mathcal{F}_t\}$ in which $\mathcal{F}_t$ is a minimal σ algebra generated by the output. The estimate, $\hat{x}$, is the estimate that comes from the Kalman filter, and furthermore, $\tilde{x}$ is the estimation error $\tilde{x} = x - \hat{x}$. The matrix K is $m \times n$ with continuous entries on the interval $[t_0, t_f]$.

The goal of this problem will be for the control to minimize a linear combination of cumulants of its cost J_1, while the disturbance wants to minimize the mean value of its cost J_2. Therefore their respective performance indices will be given by

$$\begin{aligned} \phi_1(t, x; u, w^*) &= \sum_{j=1}^{k} \gamma_j \kappa_j(t, x; u, w^*) \\ \phi_2(t, x; u^*, w) &= E\{J_2(t, x; u^*, w)\} \end{aligned} \tag{6}$$

where κ_j is the j-th cumulant of J_1, $\gamma_j \geq 0$ are constants, with $\gamma_1 > 0$, and u^*, w^* are the equilibrium solutions for the control and disturbance.

With the performance indices discussed and the problem defined, we can focus on discovering the optimal strategies for each player. First, we consider the case for the disturbance w, and then turn our attention towards the control.

A. Disturbance's Equilibrium Strategy

The approach taken to determine the disturbance's equilibrium strategy is the one taken in [8], in which completion of squares is used. Consider the matrix differential equation

$$\begin{aligned} \frac{d}{d\alpha}P(\alpha) = &-(A(\alpha) + B(\alpha)K_*(\alpha))'P(t) \\ &- P(t)(A(\alpha) + B(\alpha)K_*(\alpha)) \\ &- Q_2(\alpha) - K_*'(\alpha)R_2(\alpha)K_*(\alpha) \\ &- \frac{1}{\delta^2}P(\alpha)D(\alpha)D'(\alpha)P(\alpha) \end{aligned} \tag{7}$$

where $P(t_f) = 0$. What we would like to show is that the disturbance's equilibrium strategy is indeed

$$w_*(t) = \nu_*(t, x(t)) = -\frac{1}{\delta^2}D'(t)P(t)x(t). \tag{8}$$

Recall,

$$J_2(t_0, x_0; w, u) = E\left\{\int_{t_0}^{t_f} \left(\delta^2||w||^2 - ||z_2||^2\right) dt\right\}. \tag{9}$$

Completion of squares yields

$$\begin{aligned} J_2 = &E\left\{\int_{t_0}^{t_f} \left(\delta^2||w||^2 - ||z_2||^2 + \frac{d}{dt}x'P(t)x\right) dt\right\} \\ = &E\bigg\{\int_{t_0}^{t_f} \Big(\delta^2||w||^2 - ||z_2||^2 + \dot{x}'Px \\ &+ x'\dot{P}x + x'P\dot{x}\Big) dt\bigg\} \end{aligned} \tag{10}$$

where the dependence on time for w, z_2, and x is suppressed.

By use of (1) and (7), we obtain

$$\begin{aligned} J_2 = E\bigg\{\int_{t_0}^{t_f} \Big(&\delta^2||w||^2 - ||z_2||^2 + x'(A + BK)'Px \\ &+ x'P(A + BK)x + w'D'Px \\ &+ x'PDw - x'(A + BK_*)'Px \\ &- x'P(A + BK_*)x - x'Q_2x \\ &- x'K_*'R_2K_*x - \frac{1}{\delta^2}x'PDD'Px\Big)dt\bigg\}; \end{aligned} \tag{11}$$

and, using equation (8) for w_*, we have

$$\begin{aligned} J_2 = E\bigg\{\int_{t_0}^{t_f} \Big(&\delta^2 w'w + x'K'B'Px + x'PBKx \\ &- \delta^2 w'w_* - \delta^2 w_*'w - x'K_*'B'Px \\ &- x'PBK_*x + x'K'R_2Kx \\ &- x'K_*'R_2K_*x - \delta^2 w_*'w_*\Big)dt\bigg\}. \end{aligned} \tag{12}$$

Reducing further we find

$$\begin{aligned} J_2 = E\bigg\{\int_{t_0}^{t_f} \Big(&\delta^2||w - w_*||^2 + x'PB(K - K_*)x \\ &+ x'(K - K_*)'B'Px + x'K'R_2Kx \\ &- x'K_*'R_2K_*x\Big)dt\bigg\} \end{aligned} \tag{13}$$

so that, if the control uses its optimal gain, it is clear that the disturbance's equilibrium strategy would be w_*. Note that at this stage in the argument a specific K_* is not needed, but we will see later that the control's equilibrium solution takes the form $K_*(\alpha) = -R_1^{-1}(\alpha)B'(\alpha)\sum_{j=1}^{k}\hat{\gamma}_j\mathcal{H}_j^*(\alpha)$, in which $\mathcal{H}_j^*$ is a $n \times n$ matrix, $\hat{\gamma}_j$ is a nonnegative constant, and $\alpha \in [t_0, t_f]$.

B. Control's Equilibrium Strategy

With the disturbance's optimal strategy in place, we will now consider the control. Notice that once this strategy w_* is played, the problem becomes a k cumulant output feedback problem with $\bar{A}(t) = A(t) - \frac{1}{\delta^2}D(t)D'(t)P(t)$. With this recognition, the results of the work of Pham [9], [10] may be used.

The disturbance was able to have full state information available. However, for the control, the information available is only an estimate of the state. This, along with the linear control law assumption, yields

$$\begin{aligned} dx_a(t) = &F_a(t)x_a(t)dt + E_a(t)d\xi_a(t) \\ x_a(t_0) = &\begin{bmatrix} x_0 \\ 0 \end{bmatrix} \end{aligned} \tag{14}$$

where $x_a = (\hat{x}', \tilde{x}')'$,

$$F_a(t) = \begin{bmatrix} \bar{A}(t) + B(t)K(t) & L(t)C(t) \\ 0 & \bar{A}(t) - L(t)C(t) \end{bmatrix},$$

$$E_a(t) = \begin{bmatrix} 0 & L(t) \\ E(t) & -L(t) \end{bmatrix},$$

$$d\xi_a(t) = \begin{bmatrix} d\xi(t) \\ dv(t) \end{bmatrix},$$

and $E\{[\xi_a(t) - \xi_a(\tau)][\xi_a(t) - \xi_a(\tau)]'\} = W_a|t - \tau|$, with

$$W_a = \begin{bmatrix} W & \Gamma \\ \Gamma' & \Xi \end{bmatrix}.$$

The control's cost J_1 is now given as

$$J_1(t_0, x_0; N_a) = \int_{t_0}^{t_f} x_a'(t) N_a(t) x_a(t) dt$$

where

$$N_a(t) = \begin{bmatrix} Q_1(t) + K'(t)R_1(t)K(t) & Q_1(t) \\ Q_1(t) & Q_1(t) \end{bmatrix}.$$

With these definitions, the k-cumulant cost function is given by

$$\kappa_k = x_{a_0}' H_a(t_0; k) x_{a_0} + D(t_0; k) \tag{15}$$

where

$$\begin{aligned} \frac{d}{d\alpha} H_a(\alpha, 1) = & -F_a'(\alpha) H_a(\alpha, 1) - H_a(\alpha, 1) F_a(\alpha) \\ & -N_a(\alpha) \\ \frac{d}{d\alpha} H_a(\alpha, i) = & -F_a'(\alpha) H_a(\alpha, i) - H_a(\alpha, i) F_a(\alpha) \\ & - \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} H_a(\alpha, j) E_a(\alpha) W_a(\alpha) E_a'(\alpha) H_a(\alpha, i-j) \end{aligned} \tag{16}$$

for $i = 2, \cdots, k$ and

$$\frac{d}{d\alpha} D(\alpha, i) = -\, tr\left(H_a(\alpha, i) E_a(\alpha) W_a(\alpha) E_a'(\alpha)\right) \tag{17}$$

for $i = 1, \cdots, k$. Also, $H_a(t_f, i) = 0$ and $D_a(t_f, i) = 0$ for $1 \leq i \leq k$. Now let the symmetric matrices $H_a(\alpha)$ and $E_a(\alpha) W_a(\alpha) E_a'(\alpha)$ be partitioned as

$$H_a(\alpha, i) = \begin{bmatrix} H_{i,1}(\alpha) & H_{i,2}(\alpha) \\ H_{i,2}'(\alpha) & H_{i,3}(\alpha) \end{bmatrix} \tag{18}$$

$$E_a(\alpha) W_a(\alpha) E_a'(\alpha) = \begin{bmatrix} \Pi_1(\alpha) & \Pi_2(\alpha) \\ \Pi_2'(\alpha) & \Pi_3(\alpha) \end{bmatrix} \tag{19}$$

where

$$\begin{aligned} \Pi_1(\alpha) = & L(\alpha) \Xi L'(\alpha) \\ \Pi_2(\alpha) = & E(\alpha) \Gamma L'(\alpha) - L(\alpha) \Xi L'(\alpha) \\ \Pi_3(\alpha) = & E(\alpha) W E_a'(\alpha) - E_a(\alpha) \Gamma L'(\alpha) \\ & - L(\alpha) \Gamma' E(\alpha) - L(\alpha) \Xi L'(\alpha). \end{aligned}$$

Define the functions $\mathcal{F}_i$ and $\mathcal{G}_i$ as

$$\begin{aligned} \mathcal{F}_1(\alpha, \mathcal{H}, K) = & - [\bar{A}(\alpha) + B(\alpha)K(\alpha)]' \mathcal{H}_1(\alpha) \\ & - \mathcal{H}_1(\alpha)[\bar{A}(\alpha) + B(\alpha)K(\alpha)] \\ & - K'(\alpha) R_1(\alpha) K(\alpha) - Q_1(\alpha) \\ \mathcal{F}_i(\alpha, \mathcal{H}, K) = & - [\bar{A}(\alpha) + B(\alpha)K(\alpha)]' \mathcal{H}_i(\alpha) \\ & - \mathcal{H}_i(\alpha)[\bar{A}(\alpha) + B(\alpha)K(\alpha)] \\ & - \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} \Big[\mathcal{H}_j(\alpha) \Pi_1(\alpha) \mathcal{H}_{i-j}(\alpha) \\ & + \mathcal{H}_{k+j}(\alpha) \Pi_2(\alpha) \mathcal{H}_{i-j}(\alpha) \\ & + \mathcal{H}_j(\alpha) \Pi_2'(\alpha) \mathcal{H}_{k+i-j}(\alpha) \\ & + \mathcal{H}_{k+j}(\alpha) \Pi_3(\alpha) \mathcal{H}_{k+i-j}(\alpha) \Big] \end{aligned}$$

$$\begin{aligned} \mathcal{F}_{k+1}(\alpha, \mathcal{H}, K) = & - [\bar{A}(\alpha) + B(\alpha)K(\alpha)]' \mathcal{H}_{k+1}(\alpha) \\ & - \mathcal{H}_{k+1}(\alpha)[\bar{A}(\alpha) + L(\alpha)C(\alpha)] \\ & - \mathcal{H}_1(\alpha) L(\alpha) C(\alpha) - Q_1(\alpha) \\ \mathcal{F}_{k+i}(\alpha, \mathcal{H}, K) = & - [\bar{A}(\alpha) + B(\alpha)K(\alpha)]' \mathcal{H}_{k+i}(\alpha) \\ & - \mathcal{H}_{k+i}(\alpha)[\bar{A}(\alpha) + L(\alpha)C(\alpha)] \\ & - \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} \Big[\mathcal{H}_j(\alpha) \Pi_1(\alpha) \mathcal{H}_{k+i-j}(\alpha) \\ & + \mathcal{H}_{k+j}(\alpha) \Pi_2(\alpha) \mathcal{H}_{k+i-j}(\alpha) \\ & + \mathcal{H}_j(\alpha) \Pi_2'(\alpha) \mathcal{H}_{2k+i-j}(\alpha) \\ & + \mathcal{H}_{k+j}(\alpha) \Pi_3(\alpha) \mathcal{H}_{2k+i-j}(\alpha) \Big] \\ & - \mathcal{H}_i(\alpha) L(\alpha) C(\alpha) \end{aligned}$$

$$\begin{aligned} \mathcal{F}_{2k+1}(\alpha, \mathcal{H}, K) = & -[\bar{A}(\alpha) + L(\alpha)C(\alpha)]' \mathcal{H}_{2k+1}(\alpha) \\ & -\mathcal{H}_{2k+1}(\alpha)[\bar{A}(\alpha) + L(\alpha)C(\alpha)] \\ & -\mathcal{H}_{k+1}'(\alpha) L(\alpha) C(\alpha) \\ & -L'(\alpha) C'(\alpha) \mathcal{H}_{k+1}(\alpha) - Q_1(\alpha) \\ \mathcal{F}_{2k+i}(\alpha, \mathcal{H}, K) = & -[\bar{A}(\alpha) + B(\alpha)K(\alpha)]' \mathcal{H}_{2k+i}(\alpha) \\ & -\mathcal{H}_{k+i}(\alpha)[\bar{A}(\alpha) + B(\alpha)K(\alpha)] \\ & - \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} \Big[\mathcal{H}_{k+j}(\alpha) \Pi_1(\alpha) \mathcal{H}_{k+i-j}(\alpha) \\ & +\mathcal{H}_{2k+j}(\alpha) \Pi_2(\alpha) \mathcal{H}_{k+i-j}(\alpha) \\ & +\mathcal{H}_{k+j}(\alpha) \Pi_2'(\alpha) \mathcal{H}_{2k+i-j}(\alpha) \\ & +\mathcal{H}_{2k+j}(\alpha) \Pi_3(\alpha) \mathcal{H}_{2k+i-j}(\alpha) \Big] \\ & -\mathcal{H}_i'(\alpha) L(\alpha) C(\alpha) - C'(\alpha) L'(\alpha) \mathcal{H}_i(\alpha) \end{aligned}$$

and

$$\begin{aligned} \mathcal{G}_i(\alpha, \mathcal{H}) = & -\, tr\left(\mathcal{H}_i(\alpha) \Pi_1(\alpha)\right) - tr\left(\mathcal{H}_{k+i}(\alpha) \Pi_2(\alpha)\right) \\ & - tr\left(\mathcal{H}_{k+i}'(\alpha) \Pi_2'(\alpha)\right) - tr\left(\mathcal{H}_{2k+i}(\alpha) \Pi_3(\alpha)\right) \end{aligned}$$

where

$$\begin{aligned} \mathcal{H} = & (\mathcal{H}_1, \cdots, \mathcal{H}_k, \mathcal{H}_{k+1}, \cdots, \mathcal{H}_{2k}, \mathcal{H}_{2k+1}, \cdots, \mathcal{H}_{3k}) \\ = & (H_{1,1}, \cdots, H_{k,1}, H_{1,2}, \cdots, H_{k,2}, H_{1,3}, \cdots, H_{k,3}) \\ \mathcal{D} = & (\mathcal{D}_1, \cdots, \mathcal{D}_k) = (D_1, \cdots, D_k). \end{aligned}$$

From these definitions, it is possible to give the equations of motion,

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{H}(\alpha) =& \mathcal{F}(\alpha,\mathcal{H}(\alpha),K(\alpha))\\ \frac{d}{d\alpha}\mathcal{D}(\alpha) =& \mathcal{G}(\alpha,\mathcal{H}(\alpha))\end{aligned}$$

where $\mathcal{H}(t_f)=\mathcal{H}_f=0$ and $\mathcal{D}(t_f)=\mathcal{D}_f=0$.

Because of the quadratic nature of the cumulants, [7], we can now revise the control's performance index in (6) to

$$\begin{aligned}\hat{\phi}_1(t,\mathcal{H}(t_0,K),\mathcal{D}(t_0,K)) =& \sum_{j=1}^{k}\gamma_j\kappa_j\\ =& \sum_{j=1}^{k}\gamma_j x_0'\mathcal{H}_j(t_0,K)x_0\\ &+\gamma_j\mathcal{D}_j(t_0,K).\end{aligned}$$

With this revised performance index, a target set will also be defined. The target set is denoted as $\hat{\mathcal{M}}$ and is a closed subset of $[t_0,t_f]\times(\mathbb{R}^{n\times n})^{3k}\times\mathbb{R}^k$ such that $(t_0,\mathcal{H}_0,\mathcal{D}_0)\in\hat{\mathcal{M}}$. Furthermore the allowable gain values are from a compact set $\bar{K}\subset\mathbb{R}^{n\times m}$. The class of admissible feedback gains $\hat{\mathcal{K}}_{t_f,\mathcal{H}_f,\mathcal{D}_f;\gamma}$ are of $C([t_0,t_f];\mathcal{R}^{n\times m})$ for a finite $\hat{\phi}_1$ and for which the trajectory solutions of

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{H}(\alpha) =& \mathcal{F}(\alpha,\mathcal{H}(\alpha),K(\alpha))\\ \frac{d}{d\alpha}\mathcal{D}(\alpha) =& \mathcal{G}(\alpha,\mathcal{H}(\alpha))\end{aligned}$$

reach the target set $(t_0,\mathcal{H}_0,\mathcal{D}_0)\in\hat{\mathcal{M}}$. The control problem can be given as

$$\min_{K\in\hat{\mathcal{K}}_{t_f,\mathcal{H}_f,\mathcal{D}_f;\gamma}}\hat{\phi}(t_0,\mathcal{H}(t_0,K),\mathcal{D}(t_0,K))$$

subject to the equations of motion

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{H}(\alpha) =& \mathcal{F}(\alpha,\mathcal{H}(\alpha),K(\alpha))\\ \frac{d}{d\alpha}\mathcal{D}(\alpha) =& \mathcal{G}(\alpha,\mathcal{H}(\alpha))\end{aligned}$$

with $\mathcal{H}(t_f)=\mathcal{H}_f$ and $\mathcal{D}(t_f)=\mathcal{D}_f$.

Theorem 1 (Pham, [9], pp. 220-222): The K^* that minimizes $\hat{\phi}_1$ is given by

$$K^*(\alpha)=-R_1^{-1}(\alpha)B'(\alpha)\left[\sum_{r=1}^{k}\hat{\gamma}_r\mathcal{H}_r^*(\alpha)\right] \tag{20}$$

where $\hat{\gamma}_j=\gamma_j/\gamma_1$ and $\mathcal{H}_j$ are solutions to

$$\begin{aligned}\frac{\partial}{\partial\alpha}\mathcal{H}_1^*(\alpha) =& -[\bar{A}(\alpha)+B(\alpha)K^*(\alpha)]'\mathcal{H}_1^*(\alpha)\\ &-\mathcal{H}_1^*(\alpha)[\bar{A}(\alpha)+B(\alpha)K^*(\alpha)]\\ &-K^{*'}(\alpha)R_1(\alpha)K^*(\alpha)-Q_1(\alpha)\\ \frac{\partial}{\partial\alpha}\mathcal{H}_i^*(\alpha) =& -[\bar{A}(\alpha)+B(\alpha)K^*(\alpha)]'\mathcal{H}_i^*(\alpha)\\ &-\mathcal{H}_i^*(\alpha)[\bar{A}(\alpha)+B(\alpha)K^*(\alpha)]\\ &-\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\Big[\mathcal{H}_j^*(\alpha)\Pi_1(\alpha)\mathcal{H}_{i-j}^*(\alpha)\\ &+\mathcal{H}_{k+j}^*(\alpha)\Pi_2(\alpha)\mathcal{H}_{i-j}^*(\alpha)\\ &+\mathcal{H}_j^*(\alpha)\Pi_2'(\alpha)\mathcal{H}_{k+i-j}^*(\alpha)\\ &+\mathcal{H}_{k+j}^*(\alpha)\Pi_3(\alpha)\mathcal{H}_{k+i-j}^*(\alpha)\Big]\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{\partial}{\partial\alpha}\mathcal{H}_{k+1}^*(\alpha) =& -[\bar{A}(\alpha)+B(\alpha)K^*(\alpha)]'\mathcal{H}_{k+1}^*(\alpha)\\ &-\mathcal{H}_{k+1}^*(\alpha)[\bar{A}(\alpha)+L(\alpha)C(\alpha)]\\ &-\mathcal{H}_1^*(\alpha)L(\alpha)C(\alpha)-Q_1(\alpha)\\ \frac{\partial}{\partial\alpha}\mathcal{H}_{k+i}^*(\alpha) =& -[\bar{A}(\alpha)+B(\alpha)K^*(\alpha)]'\mathcal{H}_{k+i}^*(\alpha)\\ &-\mathcal{H}_{k+i}^*(\alpha)[\bar{A}(\alpha)+L(\alpha)C(\alpha)]\\ &-\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\Big[\mathcal{H}_j^*(\alpha)\Pi_1(\alpha)\mathcal{H}_{k+i-j}^*(\alpha)\\ &+\mathcal{H}_{k+j}^*(\alpha)\Pi_2(\alpha)\mathcal{H}_{k+i-j}^*(\alpha)\\ &+\mathcal{H}_j(\alpha)^*\Pi_2'(\alpha)\mathcal{H}_{2k+i-j}^*(\alpha)\\ &+\mathcal{H}_{k+j}^*(\alpha)\Pi_3(\alpha)\mathcal{H}_{2k+i-j}^*(\alpha)\Big]\\ &-\mathcal{H}_i^*(\alpha)L(\alpha)C(\alpha)\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{\partial}{\partial\alpha}\mathcal{H}_{2k+1}^*(\alpha) =& -[\bar{A}(\alpha)+L(\alpha)C(\alpha)]'\mathcal{H}_{2k+1}^*(\alpha)\\ &-\mathcal{H}_{2k+1}^*(\alpha)[\bar{A}(\alpha)+L(\alpha)C(\alpha)]\\ &-\mathcal{H}_{k+1}^*{}'(\alpha)L(\alpha)C(\alpha)\\ &-L'(\alpha)C'(\alpha)\mathcal{H}_{k+1}^*(\alpha)-Q_1(\alpha)\\ \frac{\partial}{\partial\alpha}\mathcal{H}_{2k+i}^*(\alpha) =& -[\bar{A}(\alpha)+L(\alpha)C(\alpha)]'\mathcal{H}_{2k+i}^*(\alpha)\\ &-\mathcal{H}_{k+i}^*(\alpha)[\bar{A}(\alpha)+L(\alpha)C(\alpha)]\\ &-\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\Big[\mathcal{H}_{k+j}^*(\alpha)\Pi_1(\alpha)\mathcal{H}_{k+i-j}^*(\alpha)\\ &+\mathcal{H}_{2k+j}^*(\alpha)\Pi_2(\alpha)\mathcal{H}_{k+i-j}^*(\alpha)\\ &+\mathcal{H}_{k+j}^*(\alpha)\Pi_2'(\alpha)\mathcal{H}_{2k+i-j}^*(\alpha)\\ &+\mathcal{H}_{2k+j}^*(\alpha)\Pi_3(\alpha)\mathcal{H}_{2k+i-j}^*(\alpha)\Big]\\ &-\mathcal{H}_i^{*'}(\alpha)L(\alpha)C(\alpha)-C'(\alpha)L'(\alpha)\mathcal{H}_i^*(\alpha)\end{aligned} \tag{23}$$

for $i=2,\cdots,k$, and $\mathcal{H}_j^*(t_f)=0$ for $j=1,\cdots,3k$. Also the Kalman gain

$$L(t)=[\Sigma(t)C'(t)+E(t)\Gamma(t)]\,\Xi^{-1} \tag{24}$$

where Σ is the solution to

$$\begin{aligned}\frac{d}{dt}\Sigma(t) =& \bar{A}(t)\Sigma(t) + \Sigma(t)\bar{A}'(t) + E(t)WE'(t) \\ &- [\Sigma(t)C'(t) + E(t)\Gamma(t)]\,\Xi^{-1} \\ &\cdot [C(t)\Sigma(t) + \Gamma'(t)E'(t)]\end{aligned} \tag{25}$$

where $\Sigma(t_0) = 0$.

IV. State Estimate Information for Both Players

In the previous section, the disturbance had full state information, while the control was required to base its decision on state estimates. It is only natural to wonder what the solution might be in the case where both players have to estimate the state. That's what will be discussed in this section.

The problem will be the same as before with a linear system described by (1) and costs (3). Before, the disturbance wished to minimize the mean of its cost, whereas the control minimized a linear combination of cumulants. In this section, the problem will be more general in which the control and disturbance will want to minimize a linear combination of cumulants. One other difference is that in this problem, both player's strategies will be assumed to be linear, that is they will be of the form

$$\begin{aligned}u(t) = \mu(t, \hat{x}(t)) =& K(t)\hat{x}(t) \\ w(t) = \nu(t, \hat{x}(t)) =& K_2(t)\hat{x}(t)\end{aligned} \tag{26}$$

where they are dependent on an estimate of the state x and K, K_2 are respectively $m \times n$ and $p \times n$ matrices with continuous entries on the interval $[t_0, t_f]$.

With this definition, we are able to define the system as before with (14), but now have

$$F_a = \begin{bmatrix} A + BK + DK_2 & LC \\ 0 & A - LC \end{bmatrix}$$

where the time arguments have been suppressed. Also, the costs can be redefined in the similar way

$$J_1(t_0, x_0; N_a) = \int_{t_0}^{t_f} x_a'(t)N_a(t)x_a(t)dt,$$

where

$$N_a(t) = \begin{bmatrix} Q_1(t) + K'(t)R_1(t)K(t) & Q_1(t) \\ Q_1(t) & Q_1(t) \end{bmatrix},$$

and

$$J_2(t_0, x_0; N_a) = \int_{t_0}^{t_f} x_a'(t)\bar{N}_a(t)x_a(t)dt,$$

where

$$\bar{N}_a = \begin{bmatrix} \delta^2 K_2'K_2 - Q_2 - K'R_2K & -Q_2 \\ -Q_2 & -Q_2 \end{bmatrix}.$$

The k-cumulant cost functions can now be given as

$$\kappa_k = x_{a_0}' H_a(t_0; k)x_{a_0} + D(t_o; k)$$

for the control and

$$\bar{\kappa}_k = x_{a_0}' \bar{H}_a(t_0; k)x_{a_0} + \bar{D}(t_o; k)$$

for the disturbance. This then gives the performance index for the control

$$\phi_1(t, x; K, K_2^*) = \sum_{j=1}^{k} x_{a_0}' H_a(t_0; k)x_{a_0} + D(t_o; k),$$

and, for the disturbance

$$\phi_2(t, x; K^*, K_2) = \sum_{j=1}^{k} x_{a_0}' \bar{H}_a(t_0; k)x_{a_0} + \bar{D}(t_o; k).$$

The functions H_a, D are still obtained by (16) and (17) $\bar{H}_a, \bar{D}$ are then given by

$$\begin{aligned}\frac{d}{d\alpha}\bar{H}_a(\alpha, 1) &= -F_a'(\alpha)\bar{H}_a(\alpha, 1) - \bar{H}_a(\alpha, 1)F_a(\alpha) \\ &\quad -N_a(\alpha) \\ \frac{d}{d\alpha}\bar{H}_a(\alpha, i) &= -F_a'(\alpha)\bar{H}_a(\alpha, i) - \bar{H}_a(\alpha, i)F_a(\alpha) \\ &- \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!}\bar{H}_a(\alpha, j)E_a(\alpha)W_a(\alpha)E_a'(\alpha)\bar{H}_a(\alpha, i-j)\end{aligned} \tag{27}$$

for $i = 2, \cdots, k$ and

$$\frac{d}{d\alpha}\bar{D}(\alpha, i) = -\,tr\left(\bar{H}_a(\alpha, i)E_a(\alpha)W_a(\alpha)E_a'(\alpha)\right) \tag{28}$$

for $i = 1, \cdots, k$. Also, recall the partition (18). Let there be a similar partition for $\bar{H}_a(\alpha, i)$. Now we can define

$$\begin{aligned}\mathcal{H} =& (\mathcal{H}_1, \cdots, \mathcal{H}_k, \mathcal{H}_{k+1}, \cdots, \mathcal{H}_{2k}, \mathcal{H}_{2k+1}, \cdots, \mathcal{H}_{3k}) \\ =& (H_{1,1}, \cdots, H_{k,1}, H_{1,2}, \cdots, H_{k,2}, H_{1,3}, \cdots, H_{k,3}), \\ \mathcal{D} =& (\mathcal{D}_1, \cdots, \mathcal{D}_k) = (D_1, \cdots, D_k).\end{aligned}$$

There are similar definitions for the disturbance, only replacing $\mathcal{H}, \mathcal{D}, \mathcal{F}, \mathcal{G}$ with $\bar{\mathcal{H}}, \bar{\mathcal{D}}, \bar{\mathcal{F}}, \bar{\mathcal{G}}$. The equations of motion for the control are then given by

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{H}(\alpha) =& \mathcal{F}(\alpha, \mathcal{H}(\alpha), K(\alpha)) \\ \frac{d}{d\alpha}\mathcal{D}(\alpha) =& \mathcal{G}(\alpha, \mathcal{H}(\alpha))\end{aligned}$$

where $\mathcal{H}(t_f) = \mathcal{H}_f$ and $\mathcal{D}(t_f) = \mathcal{D}_f$. Let $\bar{A}(\alpha) = A(\alpha) + B_2(\alpha)K_2^*(\alpha)$, then the functions $\mathcal{F}_1, \cdots, \mathcal{F}_k$ are then given as

$$\begin{aligned}\mathcal{F}_1(\alpha, \mathcal{H}, K) =& -[\bar{A}(\alpha) + B(\alpha)K(\alpha)]'\mathcal{H}_1(\alpha) \\ &- \mathcal{H}_1(\alpha)[\bar{A}(\alpha) + B(\alpha)K(\alpha)] \\ &- K'(\alpha)R_1(\alpha)K(\alpha) - Q_1(\alpha) \\ \mathcal{F}_i(\alpha, \mathcal{H}, K) =& -[\bar{A}(\alpha) + B(\alpha)K(\alpha)]'\mathcal{H}_i(\alpha) \\ &- \mathcal{H}_i(\alpha)[\bar{A}(\alpha) + B(\alpha)K(\alpha)] \\ &- \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\Big[\mathcal{H}_j(\alpha)\Pi_1(\alpha)\mathcal{H}_{i-j}(\alpha) \\ &+ \mathcal{H}_{k+j}(\alpha)\Pi_2(\alpha)\mathcal{H}_{i-j}(\alpha) \\ &+ \mathcal{H}_j(\alpha)\Pi_2'(\alpha)\mathcal{H}_{k+i-j}(\alpha) \\ &+ \mathcal{H}_{k+j}(\alpha)\Pi_3(\alpha)\mathcal{H}_{k+i-j}(\alpha)\Big]\end{aligned}$$

$$\mathcal{F}_{k+1}(\alpha,\mathcal{H},K) = -[\bar{A}(\alpha)+B(\alpha)K(\alpha)]'\mathcal{H}_{k+1}(\alpha) - \mathcal{H}_{k+1}(\alpha)[A(\alpha)+L(\alpha)C(\alpha)] - \mathcal{H}_1(\alpha)L(\alpha)C(\alpha) - Q_1(\alpha)$$

$$\begin{aligned}\mathcal{F}_{k+i}(\alpha,\mathcal{H},K) =& -[\bar{A}(\alpha)+B(\alpha)K(\alpha)]'\mathcal{H}_{k+i}(\alpha) - \mathcal{H}_{k+i}(\alpha)[A+L(\alpha)C(\alpha)] \\ & - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\Big[\mathcal{H}_j(\alpha)\Pi_1(\alpha)\mathcal{H}_{k+i-j}(\alpha) + \mathcal{H}_{k+j}(\alpha)\Pi_2(\alpha)\mathcal{H}_{k+i-j}(\alpha) \\ & + \mathcal{H}_j(\alpha)\Pi_2'(\alpha)\mathcal{H}_{2k+i-j}(\alpha) + \mathcal{H}_{k+j}(\alpha)\Pi_3(\alpha)\mathcal{H}_{2k+i-j}(\alpha)\Big] - \mathcal{H}_i(\alpha)L(\alpha)C(\alpha)\end{aligned}$$

$$\mathcal{F}_{2k+1}(\alpha,\mathcal{H},K) = -[A(\alpha)+L(\alpha)C(\alpha)]'\mathcal{H}_{2k+1}(\alpha) - \mathcal{H}_{2k+1}(\alpha)[A(\alpha)+L(\alpha)C(\alpha)] - \mathcal{H}_{k+1}'(\alpha)L(\alpha)C(\alpha) - L'(\alpha)C'(\alpha)\mathcal{H}_{k+1}(\alpha) - Q_1(\alpha)$$

$$\begin{aligned}\mathcal{F}_{2k+i}(\alpha,\mathcal{H},K) =& -[\bar{A}(\alpha)+B(\alpha)K(\alpha)]'\mathcal{H}_{2k+i}(\alpha) - \mathcal{H}_{k+i}(\alpha)[\bar{A}(\alpha)+B(\alpha)K(\alpha)] \\ & - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\Big[\mathcal{H}_{k+j}(\alpha)\Pi_1(\alpha)\mathcal{H}_{k+i-j}(\alpha) + \mathcal{H}_{2k+j}(\alpha)\Pi_2(\alpha)\mathcal{H}_{k+i-j}(\alpha) \\ & + \mathcal{H}_{k+j}(\alpha)\Pi_2'(\alpha)\mathcal{H}_{2k+i-j}(\alpha) + \mathcal{H}_{2k+j}(\alpha)\Pi_3(\alpha)\mathcal{H}_{2k+i-j}(\alpha)\Big] \\ & - \mathcal{H}_i'(\alpha)L(\alpha)C(\alpha) - C'(\alpha)L'(\alpha)\mathcal{H}_i(\alpha)\end{aligned}$$

and

$$\mathcal{G}_i(\alpha,\mathcal{H}) = -tr\left(\mathcal{H}_i(\alpha)\Pi_1(\alpha)\right) - tr\left(\mathcal{H}_{k+i}(\alpha)\Pi_2(\alpha)\right) - tr\left(\mathcal{H}_{k+i}'(\alpha)\Pi_2'(\alpha)\right) - tr\left(\mathcal{H}_{2k+i}(\alpha)\Pi_3(\alpha)\right).$$

Notice that this is almost the same as in the development for the control's strategy in the previous section. However, there is one slight difference, and that is in the $[A + LC]$ term. In the previous work, this term was affected by the disturbance's optimal strategy, because it was a full state solution. However, here, that presence is not seen.

Notice also that, for the case of $i > 1$, the result for the disturbance will be the same, only with $\mathcal{F}_j, \mathcal{H}_r$ replaced by $\bar{\mathcal{F}}_j, \bar{\mathcal{H}}_r$ and $\bar{A}(\alpha)$ replaced by $\hat{A}(\alpha) = A(\alpha) + B(\alpha)K^*(\alpha)$. This is also true for the $\mathcal{G}$ equation, for $i = 1, \cdots, k$. However for the $i = 1$ case, the $\bar{\mathcal{F}}_1$ equation becomes

$$\begin{aligned}\bar{\mathcal{F}}_1(\alpha,\bar{\mathcal{H}},K_2) =& -[\hat{A}(\alpha)+D(\alpha)K_2(\alpha)]'\bar{\mathcal{H}}_1(\alpha) - \bar{\mathcal{H}}_1(\alpha)[\hat{A}(\alpha)+D(\alpha)K_2(\alpha)] \\ & - \delta^* K_2'(\alpha)(\alpha)K(\alpha) + Q_2(\alpha) + K_2^{*'}(\alpha)R_2(\alpha)K_2(\alpha).\end{aligned}$$

Similarly the $\bar{\mathcal{F}}_{k+1}$ equation

$$\bar{\mathcal{F}}_{k+1}(\alpha,\bar{\mathcal{H}},K_2) = -[\hat{A}(\alpha)+D(\alpha)K_2(\alpha)]'\bar{\mathcal{H}}_{k+1}(\alpha) - \bar{\mathcal{H}}_{k+1}(\alpha)[A(\alpha)+L(\alpha)C(\alpha)] - \bar{\mathcal{H}}_1(\alpha)L(\alpha)C(\alpha) + Q_2(\alpha)$$

and

$$\bar{\mathcal{F}}_{2k+1}(\alpha,\mathcal{H},K_2) = -[A(\alpha)+L(\alpha)C(\alpha)]'\bar{\mathcal{H}}_{2k+1}(\alpha) - \bar{\mathcal{H}}_{2k+1}(\alpha)[A(\alpha)+L(\alpha)C(\alpha)] - \bar{\mathcal{H}}_{k+1}'(\alpha)L(\alpha)C(\alpha) - L'(\alpha)C'(\alpha)\bar{\mathcal{H}}_{k+1}(\alpha) + Q_2(\alpha)$$

for the $\bar{\mathcal{F}}_{2k+1}$ equation. The equations of motion for the disturbance are then given by

$$\frac{d}{d\alpha}\bar{\mathcal{H}}(\alpha) = \bar{\mathcal{F}}(\alpha,\bar{\mathcal{H}}(\alpha),K_2(\alpha))$$
$$\frac{d}{d\alpha}\bar{\mathcal{D}}(\alpha) = \bar{\mathcal{G}}(\alpha,\bar{\mathcal{H}}(\alpha))$$

where $\bar{\mathcal{H}}(t_f) = \bar{\mathcal{H}}_f$ and $\bar{\mathcal{D}}(t_f) = \bar{\mathcal{D}}_f$.

Theorem 2 (Output Feedback Player Strategies): The K^* that minimizes $\hat{\phi}_1$ is given by

$$K^*(\alpha) = -R_1^{-1}(\alpha)B'(\alpha)\left[\sum_{j=1}^{k}\hat{\gamma}_j\mathcal{H}_j^*(\alpha)\right] \tag{29}$$

and K_2^* that minimizes $\hat{\phi}_2$

$$K_2^*(\alpha) = -\frac{1}{\delta^2}D'(\alpha)\left[\sum_{j=1}^{k}\hat{\bar{\gamma}}_j\bar{\mathcal{H}}_j^*(\alpha)\right] \tag{30}$$

where $\hat{\gamma}_j = \gamma_j/\gamma_1$, with a similar definition for $\hat{\bar{\gamma}}_j$. The $\mathcal{H}_j$ are solutions to

$$\frac{\partial}{\partial\alpha}\mathcal{H}_1^*(\alpha) = -[\bar{A}(\alpha)+B(\alpha)K^*(\alpha)]'\mathcal{H}_1^*(\alpha) - \mathcal{H}_1^*(\alpha)[\bar{A}(\alpha)+B(\alpha)K^*(\alpha)] - K^{*'}(\alpha)R_1(\alpha)K^*(\alpha) - Q_1(\alpha) \tag{31}$$

$$\begin{aligned}\frac{\partial}{\partial\alpha}\mathcal{H}_i^*(\alpha) =& -[\bar{A}(\alpha)+B(\alpha)K^*(\alpha)]'\mathcal{H}_i^*(\alpha) - \mathcal{H}_i^*(\alpha)[\bar{A}(\alpha)+B(\alpha)K^*(\alpha)] \\ & - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\Big[\mathcal{H}_j^*(\alpha)\Pi_1(\alpha)\mathcal{H}_{i-j}^*(\alpha) + \mathcal{H}_{k+j}^*(\alpha)\Pi_2(\alpha)\mathcal{H}_{i-j}^*(\alpha) \\ & + \mathcal{H}_j^*(\alpha)\Pi_2'(\alpha)\mathcal{H}_{k+i-j}^*(\alpha) + \mathcal{H}_{k+j}^*(\alpha)\Pi_3(\alpha)\mathcal{H}_{k+i-j}^*(\alpha)\Big]\end{aligned} \tag{32}$$

$$\frac{\partial}{\partial\alpha}\mathcal{H}_{k+1}^*(\alpha) = -[\bar{A}(\alpha)+B(\alpha)K^*(\alpha)]'\mathcal{H}_{k+1}^*(\alpha) - \mathcal{H}_{k+1}^*(\alpha)[A(\alpha)+L(\alpha)C(\alpha)] - \mathcal{H}_1^*(\alpha)L(\alpha)C(\alpha) - Q_1(\alpha) \tag{33}$$

$$\frac{\partial}{\partial\alpha}\mathcal{H}^*_{k+i}(\alpha) = -[\bar{A}(\alpha) + B(\alpha)K^*(\alpha)]'\mathcal{H}^*_{k+i}(\alpha) - \mathcal{H}^*_{k+i}(\alpha)[A(\alpha) + L(\alpha)C(\alpha)] - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\Big[\mathcal{H}^*_j(\alpha)\Pi_1(\alpha)\mathcal{H}^*_{k+i-j}(\alpha) + \mathcal{H}^*_{k+j}(\alpha)\Pi_2(\alpha)\mathcal{H}^*_{k+i-j}(\alpha) + \mathcal{H}_j(\alpha)^*\Pi_2'(\alpha)\mathcal{H}^*_{2k+i-j}(\alpha) + \mathcal{H}^*_{k+j}(\alpha)\Pi_3(\alpha)\mathcal{H}^*_{2k+i-j}(\alpha)\Big] - \mathcal{H}^*_i(\alpha)L(\alpha)C(\alpha) \tag{34}$$

$$\frac{\partial}{\partial\alpha}\mathcal{H}^*_{2k+1}(\alpha) = -[A(\alpha) + L(\alpha)C(\alpha)]'\mathcal{H}^*_{2k+1}(\alpha) - \mathcal{H}^*_{2k+1}(\alpha)[A(\alpha) + L(\alpha)C(\alpha)] - \mathcal{H}^{*}_{k+1}{}'(\alpha)L(\alpha)C(\alpha) - L'(\alpha)C'(\alpha)\mathcal{H}^*_{k+1}(\alpha) - Q_1(\alpha) \tag{35}$$

$$\frac{\partial}{\partial\alpha}\mathcal{H}^*_{2k+i}(\alpha) = -[A(\alpha) + L(\alpha)C(\alpha)]'\mathcal{H}^*_{2k+i}(\alpha) - \mathcal{H}^*_{k+i}(\alpha)[A(\alpha) + L(\alpha)C(\alpha)] - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\Big[\mathcal{H}^*_{k+j}(\alpha)\Pi_1(\alpha)\mathcal{H}^*_{k+i-j}(\alpha) + \mathcal{H}^*_{2k+j}(\alpha)\Pi_2(\alpha)\mathcal{H}^*_{k+i-j}(\alpha) + \mathcal{H}^*_{k+j}(\alpha)\Pi_2'(\alpha)\mathcal{H}^*_{2k+i-j}(\alpha) + \mathcal{H}^*_{2k+j}(\alpha)\Pi_3(\alpha)\mathcal{H}^*_{2k+i-j}(\alpha)\Big] - \mathcal{H}_i^{*'}(\alpha)L(\alpha)C(\alpha) - C'(\alpha)L'(\alpha)\mathcal{H}^*_i(\alpha) \tag{36}$$

for $i = 2, \cdots, k$. Likewise, the $\bar{\mathcal{H}}_j$ are solutions to

$$\frac{\partial}{\partial\alpha}\bar{\mathcal{H}}^*_1(\alpha) = -[\hat{A}(\alpha) + D(\alpha)K_2^*(\alpha)]'\bar{\mathcal{H}}^*_1(\alpha) - \bar{\mathcal{H}}^*_1(\alpha)[\hat{A}(\alpha) + D(\alpha)K_2^*(\alpha)] - \delta^2 K_2^{*'}K_2 + K^{*'}(\alpha)R_2(\alpha)K^*(\alpha) + Q_2(\alpha) \tag{37}$$

$$\frac{\partial}{\partial\alpha}\bar{\mathcal{H}}^*_i(\alpha) = -[\hat{A}(\alpha) + D(\alpha)K_2^*(\alpha)]'\bar{\mathcal{H}}^*_i(\alpha) - \bar{\mathcal{H}}^*_i(\alpha)[\hat{A}(\alpha) + D(\alpha)K_2^*(\alpha)] - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\Big[\bar{\mathcal{H}}^*_j(\alpha)\Pi_1(\alpha)\bar{\mathcal{H}}^*_{i-j}(\alpha) + \bar{\mathcal{H}}^*_{k+j}(\alpha)\Pi_2(\alpha)\bar{\mathcal{H}}^*_{i-j}(\alpha) + \bar{\mathcal{H}}^*_j(\alpha)\Pi_2'(\alpha)\bar{\mathcal{H}}^*_{k+i-j}(\alpha) + \bar{\mathcal{H}}^*_{k+j}(\alpha)\Pi_3(\alpha)\bar{\mathcal{H}}^*_{k+i-j}(\alpha)\Big] \tag{38}$$

$$\frac{\partial}{\partial\alpha}\bar{\mathcal{H}}^*_{k+1}(\alpha) = -[\hat{A}(\alpha) + D(\alpha)K_2^*(\alpha)]'\bar{\mathcal{H}}^*_{k+1}(\alpha) - \bar{\mathcal{H}}^*_{k+1}(\alpha)[A(\alpha) + L(\alpha)C(\alpha)] - \bar{\mathcal{H}}^*_1(\alpha)L(\alpha)C(\alpha) + Q_2(\alpha) \tag{39}$$

$$\frac{\partial}{\partial\alpha}\bar{\mathcal{H}}^*_{k+i}(\alpha) = -[\hat{A}(\alpha) + D(\alpha)K_2^*(\alpha)]'\bar{\mathcal{H}}^*_{k+i}(\alpha) - \bar{\mathcal{H}}^*_{k+i}(\alpha)[A(\alpha) + L(\alpha)C(\alpha)] - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\Big[\bar{\mathcal{H}}^*_j(\alpha)\Pi_1(\alpha)\bar{\mathcal{H}}^*_{k+i-j}(\alpha) + \bar{\mathcal{H}}^*_{k+j}(\alpha)\Pi_2(\alpha)\bar{\mathcal{H}}^*_{k+i-j}(\alpha) + \bar{\mathcal{H}}(\alpha)^*\Pi_2'(\alpha)\bar{\mathcal{H}}^*_{2k+i-j}(\alpha) + \bar{\mathcal{H}}^*_{k+j}(\alpha)\Pi_3(\alpha)\bar{\mathcal{H}}^*_{2k+i-j}(\alpha)\Big] - \bar{\mathcal{H}}^*_i(\alpha)L(\alpha)C(\alpha) \tag{40}$$

$$\frac{\partial}{\partial\alpha}\bar{\mathcal{H}}^*_{2k+1}(\alpha) = -[A(\alpha) + L(\alpha)C(\alpha)]'\bar{\mathcal{H}}^*_{2k+1}(\alpha) - \bar{\mathcal{H}}^*_{2k+1}(\alpha)[A(\alpha) + L(\alpha)C(\alpha)] - \bar{\mathcal{H}}^{*}_{k+1}{}'(\alpha)L(\alpha)C(\alpha) - L'(\alpha)C'(\alpha)\bar{\mathcal{H}}^*_{k+1}(\alpha) + Q_2(\alpha) \tag{41}$$

$$\frac{\partial}{\partial\alpha}\bar{\mathcal{H}}^*_{2k+i}(\alpha) = -[A(\alpha) + L(\alpha)C(\alpha)]'\bar{\mathcal{H}}^*_{2k+i}(\alpha) - \bar{\mathcal{H}}^*_{k+i}(\alpha)[A(\alpha) + L(\alpha)C(\alpha)] - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\Big[\bar{\mathcal{H}}^*_{k+j}(\alpha)\Pi_1(\alpha)\bar{\mathcal{H}}^*_{k+i-j}(\alpha) + \bar{\mathcal{H}}^*_{2k+j}(\alpha)\Pi_2(\alpha)\bar{\mathcal{H}}^*_{k+i-j}(\alpha) + \bar{\mathcal{H}}^*_{k+j}(\alpha)\Pi_2'(\alpha)\bar{\mathcal{H}}^*_{2k+i-j}(\alpha) + \bar{\mathcal{H}}^*_{2k+j}(\alpha)\Pi_3(\alpha)\bar{\mathcal{H}}^*_{2k+i-j}(\alpha)\Big] - \bar{\mathcal{H}}_i^{*'}(\alpha)L(\alpha)C(\alpha) - C'(\alpha)L'(\alpha)\bar{\mathcal{H}}^*_i(\alpha) \tag{42}$$

for $i = 2, \cdots, k$ and $\mathcal{H}^*_j(t_f) = 0$ for $j = 1, \cdots, 3k$.

Also the Kalman gain

$$L(t) = [\Sigma(t)C'(t) + E(t)\Gamma(t)]\,\Xi^{-1} \tag{43}$$

where Σ is the solution to

$$\frac{d}{dt}\Sigma(t) = A(t)\Sigma(t) + \Sigma(t)A'(t) + E(t)WE'(t) - [\Sigma(t)C'(t) + E(t)\Gamma(t)]\,\Xi^{-1} \cdot [C(t)\Sigma(t) + \Gamma'(t)E'(t)] \tag{44}$$

where $\Sigma(t_0) = 0$.

Proof: Let K_2^* be as given in (30). Then by the use of the results in Pham, [9], pp. 220-222, the optimal K is as given in (29). Similarly if we let K^* be as in (29), then the results of Pham may be used to yield (30). ■

V. First Generation Structural Benchmark

The first generation benchmark for buildings under seismic disturbances, [12], will be used to apply the control problem where the disturbance has full state information and the control has only information on the state estimates. The uncertainty weights will be the same as was done in [3], which was taken from [6]. The parameters in the control will be given as $\delta = 19$, $\gamma_1 = 1$, $\gamma_2 = 1.1e-5$, and

$\gamma_3 = 3.2e - 13$. Along with the system description, [12] gives ten structural performance criteria, which are given as J_1-J_{10}. Note that this a bit of an abuse of notation in this paper, since we also have J_1 and J_2 as given in (3), but they are different. These criteria help determine the performance of the controlled system's interstory drifts, accelerations, and the control effort used and are not quadratic in nature. Also the benchmark paper details constraints imposed on the control effort. The control was applied to the benchmark problem and the results for the structural performance criteria are given in Table I.

TABLE I
BENCHMARK RESULTS

	LQG	3 Cumulant OF MCC
J_1	0.2898	0.2065
J_2	0.4439	0.3132
J_3	0.4843	0.7303
J_4	0.4856	0.7424
J_5	0.5976	0.7401
J_6	0.4559	0.3845
J_7	0.7096	0.6678
J_8	0.6695	1.3524
J_9	0.7807	1.3794
J_{10}	1.3142	1.5499
σ_u	0.1441	0.2369
σ_{x_m}	0.6341	0.9561
$\sigma_{\ddot{x}_{am}}$	1.0696	1.3247
$\max_t \|u\|$	0.5259	1.0151
$\max_t \|x_m\|$	2.0060	3.6412
$\max_t \|\ddot{x}_{am}\|$	4.7454	5.4388

The results show that there is a decrease of 28.7% and 29.4% of the rms drift and acceleration criteria J_1 and J_2 over LQG. Similarly, for the peak responses there is a decrease of 15.7% and 5.9% for the peak drift and acceleration criteria J_6 and J_7, respectively. Notice that the criteria that deal with the control action are higher for the output feedback MCC method. But looking at the constraints given in the problem shows that the allowed controlled effort has not been exceeded, thereby utilizing the resources present in a more efficient manner.

To show the response of the system with the control applied, plots are given in Fig. 1. The plots are given for the output feedback MCC and the uncontrolled system. The first three plots are the displacement, velocity, and acceleration of the building's floors when the earthquake history of the El Centro earthquake was applied. The floor displacement, velocity, and acceleration are greatly reduced with the aid of the control.

VI. CONCLUSION

In this paper output feedback results for the multiobjective cumulant control method were found. Two problems were examined. In the first, equilibrium solutions were given for the case in which one player has full state information, while the other player has only partial information. In the other problem, the players both had only output feedback and equilibrium solutions were determined. Finally, the results for the case of the disturbance having full state feedback were applied to the first generation structural benchmark for structures under seismic excitation.

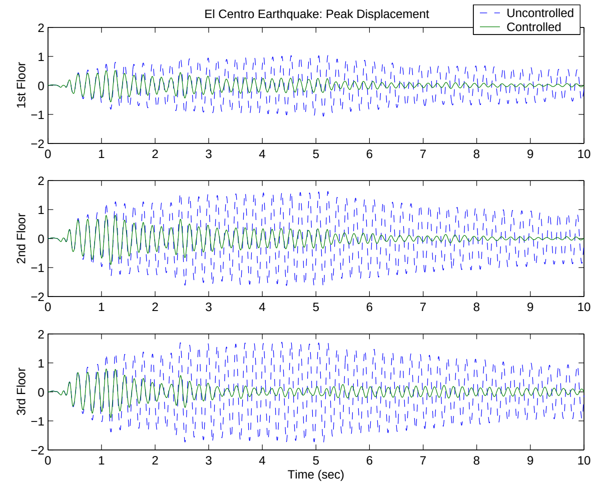

Fig. 1. Interstory Drifts for the Structure

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