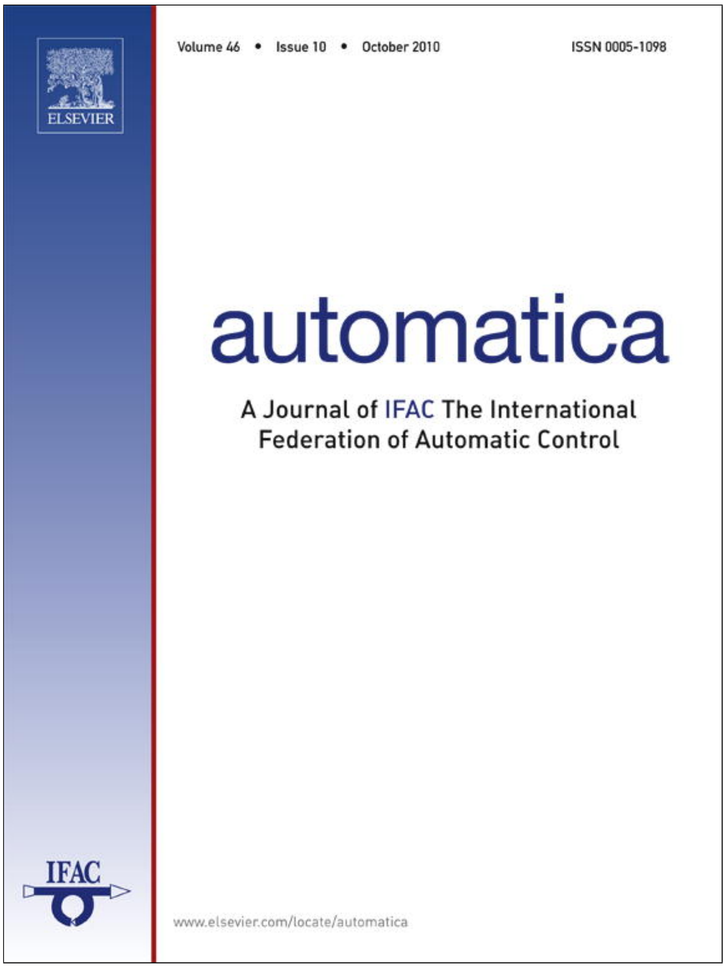
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Statistical control of control-affine nonlinear systems with nonquadratic cost functions: HJB and verification theorems[☆]

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ABSTRACT

In statistical control, the cost function is viewed as a random variable and one optimizes the distribution of the cost function through the cost cumulants. We consider a statistical control problem for a control-affine nonlinear system with a nonquadratic cost function. Using the Dynkin formula, the Hamilton–Jacobi–Bellman equation for the nth cost moment case is derived as a necessary condition for optimality and corresponding sufficient conditions are also derived. Utilizing the nth moment results, the higher order cost cumulant Hamilton–Jacobi–Bellman equations are derived. In particular, we derive HJB equations for the second, third, and fourth cost cumulants. Even though moments and cumulants are similar mathematically, in control engineering higher order cumulant control shows a greater promise in contrast to cost moment control. We present the solution for a control-affine nonlinear system using the derived Hamilton–Jacobi–Bellman equation, which we solve numerically using a neural network method.

1. Introduction

The linear-quadratic-Gaussian (LQG) optimization problem minimizes the mean, which is the first cumulant, of a quadratic cost function (Fleming & Rishel, 1975; Fleming & Soner, 1992). A more general optimization problem, however, can be formulated to shape the distribution of a cost function. This method of shaping the distribution by minimizing a cost cumulant is called statistical or cost cumulant control. Here, we optimize a control-affine nonlinear Markov diffusion process by minimizing the higher order cumulants of the nonquadratic cost function. The system is control-affine nonlinear because it is nonlinear with respect to the state variable and linear with respect to the control. Furthermore, the cost is nonquadratic with respect to the state variable, but quadratic for the control. Solutions of statistical control optimization problems are found using Hamilton–Jacobi–Bellman (HJB) equations. We derive the HJB equations for nth cost moments and cumulants. Utilizing these results, we derive necessary (HJB equations) and sufficient (verification theorems) conditions for the higher order cumulants.

To develop more intuition for statistical control, let us consider the second cumulant (variance) case. As pointed out by Mariton in Mariton (1990), the question of robustness with respect to the underlining stochastic process is important. Also, it is the performance of the sample path that we should be more concerned, and minimal mean does not consider the variance, or the distribution of the cost function, thus it does not guarantee anything about the sample path. However, the cost variance indicates to what extent the performance is spread around its mean value. This variance may play a more important role than the mean in certain applications. For example, in manufacturing, quality control is a critical profit factor, and here the ability to obtain a sharp product quality distribution curve where the mean quality can be pushed close to the rejection threshold is very important. In other words, the goal in quality control would be to have small variance of the cost function even though that may increase the mean value.

Furthermore, in seismically disturbed structural control applications, the third cumulant, skewness, is an important factor. The probability of the building failure is related to the shape of the tail end of the cost function distribution, see Spencer, Sain, Won, Kaspari, and Sain (1994). The skewness is directly related to the reliability of the structure, and it may be shaped to increase the probability of the structure surviving an earthquake. See Fig. 1. The

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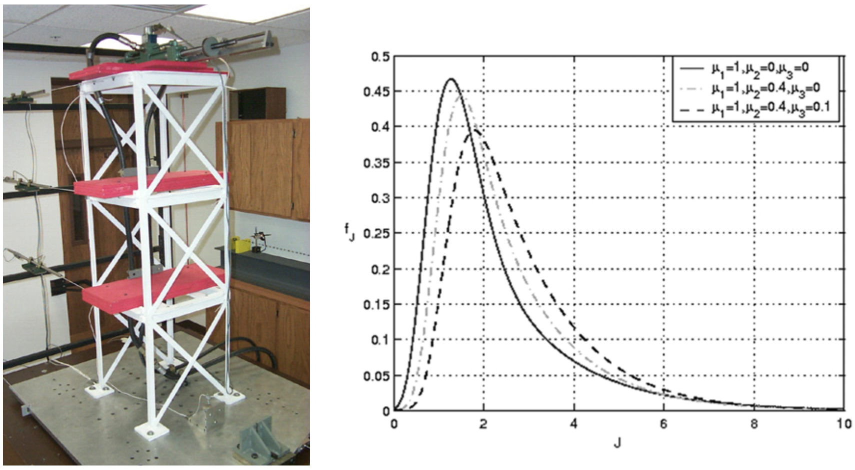

Fig. 1. Three-degree-of-freedom structure model and distribution change using statistical control.

density of the cost function changed with the first three cumulants. So, in order to shape the distribution, we use statistical control. This is the basic idea behind distribution shaping control. If the control engineer knows the desirable shape of the cost function distribution, he or she may use statistical control to achieve that objective by manipulating the cost cumulants.

In statistical control, we use cumulants because they have more intuitive meanings than moments; higher order cumulants have a decreasing significance which leads to better approximation scheme; and it leads to a linear controller for the linear quadratic case even for higher order cumulant optimization. Classical LQG control is a special case of statistical control when the first cumulant (mean) is optimized. Even for a linear system and a quadratic cost function, the minimization of the first two moments leads to a nonlinear controller as shown in Won (2005).

Another special case of statistical control is risk-sensitive (RS) control where it uses all denumerable sums of the quadratic cost function. However, the coefficients of these cumulants are fixed by the risk-sensitivity parameter. RS control lacks the flexibility to control individual cost cumulants.

Statistical control has a relatively long history going back to the sixties. An open-loop minimum cost variance problem, which is the second cumulant, was solved in 1966 (Sain, 1996). Liberty and Hartwig studied the quadratic nature of the minimal statistical control problem in Liberty and Hartwig (1976). However, their study was restricted to a linear system and a quadratic cost. The relationship between second order cumulant statistical control and risk-sensitive control was investigated in Won (2005). We published results for second cost cumulant control in Sain, Won, Spencer, and Liberty (2000). There, we only derived the HJB equation for the second cumulant case. Here, we derive the HJB equation for the nth cumulant case. Even though the problem formulation is for a general nonlinear system, we only solved the optimal control problem for a linear system with quadratic cost function in Sain et al. (2000). In this paper, we extend this result to nth cumulant optimization for a control-affine nonlinear system with a nonquadratic cost function. In Won (2006), only the cost moment case was discussed and not the cost cumulant case. This paper is an evolution of the preliminary results presented in Won (2005), where no verification theorems were given. The deterministic version of a control-affine, a nonquadratic cost function problem was solved in Won and Biswas (2007).

Statistical control has been applied to structural vibration control. In the control of structures, benchmark problems have been developed for testing different control algorithms (see for example Spencer, Dyke, and Deoskar (1998) and Yang, Agrawal, Samali, and Wu (2004)). In Pham, Sain, and Liberty (2002) and Pham (2005), cumulant control is applied to buildings and bridges under seismic excitation. It has also been applied to the satellite attitude control problem (Lee, Diersing, & Won, 2008). However, in those works, we assumed a linear system which restricted the solution for specified linear regions. This gave us the motivation to study the nonlinear case presented in this paper.

In the next section, we state the optimal statistical control problem. In Section 3, we derive the HJB Equation for the nth cost moment, which is a necessary condition for optimality. We do not use the traditional approach of using Bellman's principle of optimality to derive the HJB equations, but instead we use Dynkin's formula. We also prove the sufficient condition, the verification theorem, for optimality in this section. In Section 4, we describe a procedure to generate the necessary and sufficient conditions for nth cumulant case. Then we use this procedure to derive second, third, and fourth cumulant HJB equations and verification theorems. As an example, we provide the solution of first and second cumulant optimization of a control-affine nonlinear system in Section 5. Finally, conclusions are given in the last section.

2. Optimal statistical control problem formulation

Let $Q_0 = [t_0, t_F) \times \mathbb{R}^n$, $\bar{Q}_0$ denote the closure of Q_0, $T = [t_0, t_F]$, and let $U \subset \mathbb{R}^m$ denote a set from which a control applied at any time t is chosen. Consider a nonlinear stochastic differential equation:

$$\mathrm{d}x(t) = f(t, x(t), u(t))\mathrm{d}t + \sigma(t, x(t))\mathrm{d}w(t), \tag{1}$$

where $t \in T$, $x(t_0) = x_0$, and $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in U$ is the control action, and $\mathrm{d}w(t)$ is a Gaussian random process of dimension d with zero mean and covariance of $W(t)\mathrm{d}t$. Assume $f : \bar{Q}_0 \times U \to \mathbb{R}^n$ is $C^1(\bar{Q}_0 \times U)$, and $\sigma : \bar{Q}_0 \to \mathbb{R}^{n\times d}$ is $C^1(\bar{Q}_0)$. We also assume that f and σ satisfy Lipschitz and linear growth conditions; see Arnold (1974, p. 113) and Gardiner (1997, p. 94) for detailed discussion on these conditions.

A memoryless feedback *control law* is introduced as $u(t) = k(t, x(t))$, $t \in T$, where k is a nonrandom function with random arguments. Now we admit only the bounded, Borel measurable feedback control law, $k(t, x) : \bar{Q}_0 \to U$ such that $k(t, x)$ satisfies a local Lipschitz condition and a linear growth condition. A feedback control law that satisfies both of these conditions is called *admissible*. Then a pathwise unique solution process $x(t)$ of (1) exists in probability, see Fleming and Soner (1992, p. 159) and Wong and Hajek (1985). Consider a nonquadratic cost-to-go function,

$$J(t, x(t), k) = \int_t^{t_F} \Big[L(s, x(s), k(s, x(s))) \Big] \mathrm{d}s + \psi(x(t_F)), \tag{2}$$

where $L : \bar{Q}_0 \times U \to \mathbb{R}^+$ is $C(\bar{Q}_0 \times U)$ and $\psi : \mathbb{R}^n \to \mathbb{R}^+$ is $C(\mathbb{R}^n)$. Also assume that L and ψ satisfy polynomial growth conditions. Fleming and Rishel showed that a process $x(t)$ from (1), having an admissible controller k, together with polynomial growth conditions ensure that $E\{J(t, x(t), k)|x(t) = x\}$ is finite, see Fleming and Rishel (1975, p. 157). Furthermore, we utilize the backward evolution operator, $\mathcal{O}(k)$, as defined in Fleming and Soner (1992) and Sain et al. (2000):

$$\begin{aligned}\mathcal{O}(k) &= \frac{\partial}{\partial t} + f'(t, x, k(t, x))\frac{\partial}{\partial x} \\ &\quad + \frac{1}{2}\mathrm{tr}\left(\sigma(t, x)W(t)\sigma'(t, x)\frac{\partial^2}{\partial x^2}\right) \\ &\triangleq \frac{\partial}{\partial t} + \mathcal{O}^{(1)}(k) + \mathcal{O}^{(2)}. \qquad (3)\end{aligned}$$

See also Fleming and Soner (1992, page 121) for more detailed definition of the backward evolution operator. We introduce the Dynkin's formula. See Fleming and Soner (1992, pages 121, 135, 161). Here, we use the shortened notation E_{tx} for the conditional expectation $E\{\cdot|x(t) = x\}$.

$$\Phi(t, x) = E_{tx}\left\{\int_t^{t_F} -\mathcal{O}(k)\Phi(s, x(s))\,\mathrm{d}s + \Phi(t_F, x(t_F))\right\}. \qquad (4)$$

Assumptions $\Phi(t, x) \in C_p^{1,2}(\bar{Q}_0)$, k admissible, and $E\{|x(s)|^m |x(t) = x\}$ bounded insure that the terms in the right hand side of (4) are finite.

Moments are defined as $M_i(t, x, k) = E\left\{J^i(t, x, k)|x(t) = x\right\}$. $M_0 = 1$ by definition. Moments are a set of descriptive constants of a distribution that are useful for measuring its properties. Formally, cumulants $V_1, V_2, \ldots, V_i$ are defined by the identity in t (Stuart & Ord, 1987, p. 84), $\exp\left(\sum_{i=1}^{\infty} V_i \frac{t^i}{i!}\right) = \sum_{i=0}^{\infty} M_i \frac{t^i}{i!}$. The subscript, i, denotes a fixed positive integer. Note that V_0 is not defined. Moreover, we have the following moment and cumulant generating functions,

$$\phi(t) = \int_{-\infty}^{\infty} \exp^{\mathrm{j}tx}\,\mathrm{d}F \quad \text{and} \quad \Xi(t) = \log\phi(t),$$

where j is the complex operator, $\mathrm{j}^2 = -1$.

The following definitions are used to define the statistical control problem.

Definition 1. A function $M_1 : \bar{Q}_0 \to \mathbb{R}^+$, which is $C^{1,2}(\bar{Q}_0)$, is an *admissible mean cost function* if there exists an admissible control law k such that $V_1(t, x, k) = M_1(t, x)$ for $t \in T$ and $x \in \mathbb{R}^n$.

Here, we solve the mean cost constraint equation for all possible solutions and then minimize the cost variance within those possible solutions.

Consider an open set, $Q \subset Q_0$. We will need this Q for the verification theorems. We use the notation, $C^{1,2}(Q)$ to denote the space of functions continuous on Q together with the partial derivatives. Similarly, $C_p^{1,2}(Q)$ denotes a class of functions in $C^{1,2}(Q)$ which satisfy a polynomial growth condition on Q. The notation $C_p^{1,2}(Q) \cap C(\bar{Q})$ denotes that the function is in $C_p^{1,2}(Q)$ and continuous on the closure of Q. See Fleming and Rishel (1975, p. 124) for the detailed explanation of these notations.

Definition 2. Every admissible M_1 defines a class K_{M_1} of *control laws k corresponding to* M_1 in the manner that $k \in K_{M_1}$, if k is an admissible control law which satisfies the previous definition.

Definition 3. Let M_1 be an admissible mean cost function, and let K_{M_1} be its induced class of admissible control laws. A *minimal statistical control law*, $k^*_{V_i|M_1}$ satisfies $V_i(t, x, k^*_{V_i|M_1}) = V_i^*(t, x) \le V_i(t, x, k)$, for $t \in T$, $x \in \mathbb{R}^n$, whenever $k \in K_{M_1}$. The subscript, i denotes a fixed non-negative integer.

Definition 4. Similarly, we define the *minimal* ith *moment control law*, $k^*_{M_i|M_1}$ as $M_i(t, x, k^*_{M_i|M_1}) = M_i^*(t, x) \le M_i(t, x, k)$.

The statistical control problem assumes that a mean cost $M_1(t, x)$ has been specified, and it seeks a control law which minimizes the other cost cumulant. For example, for the second cumulant case, we find all controllers that give a pre-specified admissible mean (first cumulant) and within this set we find the controller that minimizes the variance (second cumulants). In a sense, we deal with one objective at a time. The admissible cost cumulant function does not have to be the first cumulant (mean). It can be the second or any other cost cumulant. A general *statistical control problem* seeks a control law which minimizes the nth cost cumulant while keeping another admissible cost cumulant at a pre-specified level. This is a key to finding an optimal solution. We note that the controls that make M_{i-1} admissible are not necessarily disjoint nor overlapping with M_i.

In this paper, we assume that the value functions are twice continuously differentiable. In fact, we only need to assume a uniformly parabolic HJB equation for the existence of the unique twice differentiable value function (Fleming & Soner, 1992, p. 168). However if the HJB equation is a degenerate parabolic type then a smooth solution, $V(t, x)$ cannot be expected. In this paper, as an initial investigation of the nonlinear statistical control theory, we will make the differentiability assumptions. We will investigate the use of viscosity solutions in future research for nondifferentiable value functions.

3. *n*th moment HJB equation

We derive the nth moment Hamilton–Jacobi–Bellman (HJB) equation assuming that a sufficiently smooth solution exists. The first moment and second cumulant HJB equations and associated verification theorems are derived in Sain et al. (2000). We do not directly use Bellman's principle of optimality (dynamic programing principle) in deriving this HJB equation. This HJB equation is a necessary condition for the optimality, and we utilize this result to derive the nth cumulant HJB equations. We derive the HJB equation for the nth moment case. We show that if the optimal controller exists, then it has to satisfy the derived HJB equation.

Theorem 3.1. *Assume $M_i^*(t, x) \in C_p^{1,2}(\bar{Q}_0)$ and the existence of an optimal controller $k^*_{M_i|M_1}$, where i denotes a fixed non-negative integer. By definition, M_0 is an identity. Then $k^*_{M_i|M_1}$ and $M_i^*(t, x)$ satisfy the partial differential equation*

$$\mathcal{O}(k^*_{M_i|M_1})[M_i^*(t, x)] + iM^*_{i-1}(t, x)L(t, x, k^*_{M_i|M_1}) = 0 \qquad (5)$$

for $t \in T$, $x \in \mathbb{R}^n$, where

$$\begin{aligned}&\mathcal{O}(k^*_{M_i|M_1})[M_i^*(t, x)] + iM^*_{i-1}(t, x)L(t, x, k^*_{M_i|M_1}) \\ &\quad = \min_{k \in K_{M_1}} \left\{\mathcal{O}(k)[M_i^*(t, x)] + iM^*_{i-1}(t, x)L(t, x, k)\right\}, \qquad (6)\end{aligned}$$

along with the boundary condition $M_i^*(t_F, x) = \psi^i(x(t_F))$, $x \in \mathbb{R}^n$.

Proof. Using the algebraic identity,

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \qquad (7)$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, with $J_a = \int_t^{t+\Delta t} L_s \mathrm{d}s$, and $J_b = \int_{t+\Delta t}^{t_F} L^* \mathrm{d}s + \psi(x^*(t_F))$, we obtain $[J(t, x(t), k_1)]^i = [J_a + J_b]^i = J_b^i + iJ_a J_b^{i-1} + \sum_{k=2}^{i} \frac{i!}{k!(i-k)!} J_a^k J_b^{i-k}$.

Define a controller $k_1 \in K_{M_1}$ by the action, $k_1(r, x) = k(r, x)$, if $t \le r \le t + \Delta t$, and $k_1(r, x) = k^*_{M_i|M_1}(r, x)$, if $t + \Delta t < r \le t_F$. Then the ith moment is given by $M_i(t, x(t); k_1) = E_{tx}\{J^i(t, x(t), k_1)\}$. Using the algebraic identity with $J_a = \int_t^{t+\Delta t} L_s \mathrm{d}s$ and $J_b = \int_{t+\Delta t}^{t_F} L^* \mathrm{d}s + \psi(x^*(t_F))$, we have $[J(t, x(t), k_1)]^i = [J_a + J_b]^i = J_b^i + iJ_a J_b^{i-1} + \sum_{k=2}^{i} \frac{i!}{k!(i-k)!} J_a^k J_b^{i-k}$. By definition $M_i^*(t, x) \le M_i(t, x; k_1)$. Substitute $M_i(t, x; k_1)$ and obtain

$$M_i^*(t, x) \le E_{tx}\left\{ J_b^i + iJ_a J_b^{i-1} + \sum_{k=2}^{i} \frac{i!}{k!(i-k)!} J_a^k J_b^{i-k} \right\}. \tag{8}$$

The first term on the right side of the inequality can be rewritten as

$E_{tx}\left\{J_b^i\right\} = E_{tx}\left\{M_i^*(t + \Delta t, x(t + \Delta t))\right\}$. Use the Dynkin Formula (4) to obtain,

$$E_{tx}\left\{M_i^*(t + \Delta t, x(t + \Delta t))\right\} - M_i^*(t, x) = E_{tx}\left\{ \int_t^{t+\Delta t} \mathcal{O}(k)[M_i^*(r, x(r))]\mathrm{d}r \right\}.$$

Consequently, we obtain

$$E_{tx}\left\{J_b^i\right\} = M_i^*(t, x) + \Delta t E_{tx}\left\{\mathcal{O}(k)[M_i^*(t^+, x(t^+))]\right\}. \tag{9}$$

The second term on the right side of the inequality in (8) can be rewritten as

$$E_{tx}\left\{iJ_a^i J_b^{i-1}\right\} = E_{tx}\left\{ i\int_t^{t+\Delta t} L_s \mathrm{d}s \left[\int_{t+\Delta t}^{t_F} L^* \mathrm{d}s + \psi(x^*(t_F)) \right]^{i-1} \right\},$$

where $L_s = L(s, x(s), k(s, x(s)))$, $L^* = L(s, x^*(s), k^*_{M_i|M_1}(s, x^*(s)))$, and $x^*(t)$ is the solution of (1) when $k = k^*_{M_i|M_1}$. Using the mean value theorem, we have

$$E_{tx}\left\{iJ_a^i J_b^{i-1}\right\} = E_{tx}\left\{ i\Delta t L\left(t^+, x^+, k^+\right) \times \left[\int_{t+\Delta t}^{t_F} L^* \mathrm{d}s + \psi\left(x^*\left(t_F\right)\right) \right]^{i-1} \right\}, \tag{10}$$

where $t \le t^+ \le t + \Delta t$. The third term on the right side of the inequality in (8) can be rewritten as

$$E_{tx}\left\{ \sum_{k=2}^{i} \frac{i!}{k!(i-k)!} \left[\Delta t^k L^k(t^+, x^+, k^+)\right] \times \left[\int_{t+\Delta t}^{t_F} L^* \mathrm{d}s + \psi(x^*(t_F)) \right]^{i-k} \right\}. \tag{11}$$

Substitute (9)–(11) into (8) and obtain

$$\begin{aligned} M_i^*(t, x) \le{}& M_i^*(t, x) + \Delta t E_{tx}\left\{\mathcal{O}(k)\left[M_i^*\left(t^+, x\left(t^+\right)\right)\right]\right\} \\ &+ E_{tx}\left\{ i\Delta t L\left(t^+, x^+, k^+\right) \times \left[\int_{t+\Delta t}^{t_F} L^* \mathrm{d}s + \psi\left(x^*\left(t_F\right)\right) \right]^{i-1} \right\} \\ &+ E_{tx}\left\{ \sum_{k=2}^{i} \frac{i!}{k!\,(k-1)!} \left[\Delta t^k L^k\left(t^+, x^+, k^+\right)\right] \times \left[\int_{t+\Delta t}^{t_F} L^* \mathrm{d}s + \psi\left(x^*\left(t_F\right)\right) \right]^{i-k} \right\}. \end{aligned}$$

Using the uniform integrability condition (for example see Sain et al. (2000, Lemma 4.3)), when we divide by Δt and let $\Delta t \to 0$, we obtain $0 \le \mathcal{O}(k)[M_i^*(t, x)] + iL(t, x, k)M_{i-1}^*(t, x)$. The inequality becomes an equality when $k(t, x) = k^*_{M_i/M_1}(t, x)$. □

This result is critical in developing the nth cumulant optimization HJB equation, which we will show in the next section.

The key assumption needed for the unique existence of M^* in (6) is uniform parabolicity of the HJB equations, see Fleming and Soner (1992, p. 162). The nth moment HJB equation is still a second order nonlinear partial differential equation. It differs from the traditional first cumulant (mean) HJB equation by iM_{i-1}^* in front of L in (6). So we need to show that the Hamiltonian with iM_{i-1}^*L satisfies Lipschitz and linear growth condition. The existence of M^* leads to existence of a Borel measurable optimal controller by the selection theorem (Fleming & Rishel, 1975, p. 170).

Remarks. In traditional derivation of the HJB equation using Bellman's optimality principle (dynamic programing principle), one requires the additivity of the cost function and the strong Markov assumptions. The higher order moments cost function is not additive so, we cannot derive nth moment HJB equation following the traditional method such as the one in Yong and Zhou (1999).

Before moving on to the accompanying sufficient condition, the following useful lemma will be presented.

Lemma 3.1. *Consider the running cost function $L(t, x, k(t, x))$, which is denoted by L_t. then the equality*

$$(j+1)\int_t^{t_f} L_s \left[\int_s^{t_f} L_r \mathrm{d}r \right]^j \mathrm{d}s = \left[\int_t^{t_f} L_r \mathrm{d}r \right]^{j+1} \tag{12}$$

holds.

Proof. First we should change the limits of integration:

$$\int_t^{t_f} L_s \left[\int_s^{t_f} L_r \mathrm{d}r \right]^j \mathrm{d}s = (-1)^j \int_{t_f}^{t} L_s \left[\int_{t_f}^{s} L_r \mathrm{d}r \right]^j \mathrm{d}s.$$

Now recall that for two differential functions F and G we can integrate by parts

$$\int_{t_f}^{t} F(s)g(s)\mathrm{d}s = F(t)G(t) - F(t_f)G(t_f) - \int_{t_f}^{t} f(s)G(s)\mathrm{d}s$$

where $f(s) = \frac{\mathrm{d}F(s)}{\mathrm{d}s}$, $G(s) = \int_{t_f}^{s} g(r)\mathrm{d}r$. Let $g(s) = L_s$ and $F(s) = [\int_{t_f}^{s} L_r \mathrm{d}r]^j$. With these definitions, we note that $f(s) = jL_s[\int_{t_f}^{s} L_r \mathrm{d}r]^{j-1}$ and $G(s) = \int_{t_f}^{s} L_r \mathrm{d}r$, which then yields

$$(-1)^j \int_{t_f}^{t} L_s \left[\int_{t_f}^{s} L_r \mathrm{d}r \right]^j \mathrm{d}s = (-1)^j \left[\int_{t_f}^{t} L_s \mathrm{d}s \right]^{(j+1)} - (-1)^j \int_{t_f}^{t} jL_s \left[\int_{t_f}^{s} L_r \mathrm{d}r \right]^j \mathrm{d}s$$

which is

$$(j+1)\int_{t_f}^{t} L_s \left[\int_{t_f}^{s} L_r \mathrm{d}r \right]^j \mathrm{d}s = \left[\int_{t_f}^{t} L_s \mathrm{d}s \right]^{(j+1)}$$

and the lemma is proved. □

Now consider the jth moment equation. We can show that a function M_j^* that satisfies this equation is in fact the jth moment. Before beginning, we define an admissible ith moment cost function as M_i^* if there exists a control law such that $M_i^*(t, x) = M_i(t, x; k)$.

Lemma 3.2. *Consider a function $M_j^* \in C_p^{1,2}(Q) \cap C(\bar{Q})$ that satisfies*

$$\mathcal{O}(k)M_j^*(t,x) + jM_{j-1}^*(t,x)L(t,x,k(t,x)) = 0 \tag{13}$$

where M_{j-1}^* is an admissible $(j-1)$ moment cost function; then $M_j^*(t,x) = M_j(t,x;k)$.

Proof. Suppose that M_{j-1}^* is indeed an admissible $(j-1)$ cost function and that M_j^* satisfies (13). Since $M_j^* \in C_p^{1,2}(Q)$, the Dynkin formula (4) can be used to give

$$M_j^* = E_{tx}\left\{\int_t^{t_f} jM_{j-1}^*(s,x(s))L_s \mathrm{d}s + \psi^j(x(t_f))\right\} \tag{14}$$

where $L(t,x,k(t,x))$ is denoted by L_t. But if M_{j-1}^* is an admissible $(j-1)$ cost function, then it is such that $M_{j-1}^* = M_{j-1}(t,x;k)$. Therefore, we have

$$M_j^* = E_{tx}\left\{\int_t^{t_f} jL_s E_{sx}\left\{\left[\int_s^{t_f} L_r \mathrm{d}r + \psi(x(t_f))\right]^{j-1}\right\}\mathrm{d}s + \psi^j(x(t_f))\right\} \tag{15}$$

which gives

$$\begin{aligned} M_j^* &= E_{tx}\left\{l\int_t^{t_f} E_{sx}\left\{jL_s\left[\int_s^{t_f} L_r \mathrm{d}r + \psi(x(t_f))\right]^{j-1}\right\}\mathrm{d}s + \psi^j(x(t_f))\right\} \\ &= E_{tx}\left\{E_{sx}\left\{\int_t^{t_f} jL_s\left[\int_s^{t_f} L_r \mathrm{d}r + \psi(x(t_f))\right]^{j-1}\mathrm{d}s\right\}\right\} + E_{tx}\left\{\psi^j(x(t_f))\right\} \\ &= E_{tx}\left\{\int_t^{t_f} jL_s\left[\int_s^{t_f} L_r \mathrm{d}r + \psi(x(t_f))\right]^{j-1}\mathrm{d}s + \psi^j(x(t_f))\right\}. \end{aligned}$$

The binomial formula (7) can now be applied to the term in the integral that is raised to the $(j-1)$st power where $p = \int_s^{t_f} L_r \mathrm{d}r$ and $q = \psi(x(t_f))$. This yields

$$\left[\int_s^{t_f} L_r \mathrm{d}r + \psi(x(t_f))\right]^{j-1} = \sum_{i=0}^{j-1}\binom{j-1}{i}\left[\int_s^{t_f} L_r \mathrm{d}r\right]^{j-1-i}\psi^m(x(t_f));$$

and, by looking at the term in the expectation, we have

$$\begin{aligned} &\int_t^{t_f} jL_s\sum_{i=0}^{j-1}\binom{j-1}{i}\left[\int_s^{t_f} L_r \mathrm{d}r\right]^{j-1-i}\psi^i(x(t_f))\mathrm{d}s + \psi^j(x(t_f)) \\ &= \sum_{i=0}^{j-1}\binom{j-1}{i}\int_t^{t_f} jL_s\left[\int_s^{t_f} L_r \mathrm{d}r\right]^{j-1-i} \times \psi^i(x(t_f))\mathrm{d}s + \psi^j(x(t_f)). \end{aligned}$$

But notice that $j\binom{j-1}{i} = j\frac{(j-1)!}{(j-1-i)!i!} = \frac{j!}{(j-i)!i!}(j-i)$. This results in

$$\sum_{i=0}^{j-1}\binom{j}{i}\int_t^{t_f}(j-i)L_s\left[\int_s^{t_f} L_r \mathrm{d}r\right]^{j-1-i}\psi^i(x(t_f))\mathrm{d}s + \psi^j(x(t_f));$$

and, by Lemma 3.1, we have

$$\sum_{i=0}^{j-1}\binom{j}{i}\left[\int_t^{t_f} L_s \mathrm{d}s\right]^{j-i}\psi^i(x(t_f))\mathrm{d}s + \psi^j(x(t_f)) = \left[\int_t^{t_f} L_s \mathrm{d}s + \psi(x(t_f))\right]^j.$$

Thus $M_j^*(t,x) = M_j(t,x;k)$ for k admissible and the lemma is proved. □

Now consider the equation

$$\min_{k\in K_{M_1}}\{\mathcal{O}(k)M_{j+1}^* + (j+1)M_jL(t,x,k)\} = 0 \tag{16}$$

where $M_{j+1}^*(t,x)$ is a suitably smooth solution to (16) and K_{M_1} is the class of admissible control laws. Suppose that the moment that is desired to be minimized is the $(j+1)$-st moment.

We assume that a sufficiently smooth solution to the HJB equation exists for the $(j+1)$th moment, $M_{j+1}^*(t,x)$. Then, the verification for the nth moment states that it is the optimal cost of control and by using this fact, we can find the optimal control law.

Theorem 3.2 (*nth Moment Verification Theorem*). *Let $M_j^* \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be the jth admissible moment cost function with an admissible class of control strategies, K_{M_1}. If the function $M_{j+1}^* \in C_p^{1,2}(Q) \cap C(\bar{Q})$ satisfies*

$$\min_{k\in K_{M_1}}\left\{\mathcal{O}(k)M_{j+1}^*(t,x) + (j+1)M_j^*(t,x)L(t,x,k(t,x))\right\} = 0, \tag{17}$$

then $M_{j+1}^*(t,x) \le M_{j+1}(t,x;k)$ for all $k \in K_{M_1}$ and $(t,x) \in Q$. Furthermore if there is a $k_{M_i|M_1}^*$ such that

$$k_{M_i|M_1}^* = \arg\min_{k\in K_{M_1}}\left\{\mathcal{O}(k)M_{j+1}^*(t,x) + (j+1)M_j^*(t,x)L(t,x,k(t,x))\right\} \tag{18}$$

then $M_{j+1}^*(t,x) = M_{j+1}(t,x,k_{M_i|M_1}^*)$.

Proof. The proof is the same as that of Lemma 3.2 except that the equality sign is now an inequality

$$M_j^*(t,x) \le E_{tx}\left\{\int_t^{t_f} jM_{j-1}^*(s,x(s))L_s \mathrm{d}s + \psi^j(x(t_f))\right\}.$$

Beyond this, the proof is similar to the previous case. For the case of $k = k_{M_i|M_1}^*$, the proof is similar to the proof in Lemma 3.2. □

Remarks. Here, we note that minimal variance control and cost moment control are present in the literature, but not statistical (cost cumulant) control. Minimum variance control, which minimizes the variance of the output, has not been very successful. This phrase, minimal variance control, was coined by Karl Astrom and his colleagues. It is not cost variance, but rather output variance. From an engineering point of view, Astrom's variance control is very similar to LQG. $E\{x'x\} = tr\, E\{xx'\}$ As such, we may lump it together with the work of Kalman. Therefore, it is in essence the same idea as cost average control.

Remarks. Minimal cost moment control also exists, however, this is different from cost cumulant control (Sain, 1967). Even though from a mathematician's point of view, cost moments may be similar to cost cumulants, they give very different results from a control engineer's point of view. For example, in cost moment minimization, a nonlinear controller may result from a linear system, quadratic cost case (Won, 2005). However cost cumulant control gives a linear controller. Moreover, the control of higher moments is problematic. If we control just the first few of them, the neglected higher moments may have more effect than the ones that we have chosen to control. This makes moment control very sensitive to modeling errors. This is not desirable in theory, in computation, in approximation, or in application. With cumulants, controlling the first few is an excellent approximation to controlling them all, as the neglected ones tend to produce only smaller and smaller effects.

4. Higher order cumulant HJB equations and verification theorems

Here, we derive the HJB equations for second, third, and fourth cost cumulant statistical control. Cumulants can be generated from a logarithmic transformation of the moment generating function. The logarithmic transformation corresponds to a change of probability measure, and it is useful in studying certain asymptotic problems. We will utilize the moment–cumulant relationship and the nth moment HJB equation to find the nth cumulant HJB equation. Then we derive the verification theorems for the second, third, and fourth cost cumulant control.

Lemma 4.1. *The ith cost moment, $M_i(t, x; k)$ is related to the (i-1)th cost moment, $M_{i-1}(t, x; k)$ by the following partial differential equation.*

$$\mathcal{O}(k)[M_i(t,x;k)] + iM_{i-1}(t,x;k)L(t,x,k) = 0 \tag{19}$$

with the boundary condition $M_i(t_f, x; k) = \psi^i(x(t_F))$ where $i = 1, 2, \ldots$.

Proof. This is from Theorem 3.1. □

We note that in the above lemma M_i is not an optimal cost function. We also note that the first, second, and third moment HJB equations are given respectively as $0 = \mathcal{O}(k)M_1 + L$, $0 = \mathcal{O}(k)M_2 + 2M_1L$, and $0 = \mathcal{O}(k)M_3 + 3M_2L$.

Lemma 4.2. *The powers of cost moments M_i are related by the following partial differential equation.*

$$\mathcal{O}(k)[M_i^p(t,x;k)] = -piM_i^{p-1}(t,x;k)L(t,x,k) + \frac{p(p-1)}{2}M_i^{p-2}(t,x;k)\left\|\frac{\partial M_i(t,x;k)}{\partial x}\right\|^2_{\sigma W\sigma'}. \tag{20}$$

Proof. By definition, we have

$$\mathcal{O}(k)[M_i^p] = pM_i^{p-1}\mathcal{O}^{(1)}(k)[M_i] + \mathcal{O}^{(2)}[M_i^p] + pM_i^{p-1}\frac{\partial M_i}{\partial t}. \tag{21}$$

From Theorem 3.1, we obtain $\mathcal{O}^{(1)}(k)[M_i] + \mathcal{O}^{(2)}[M_i] + \frac{\partial M_i}{\partial t} + iM_{i-1}L = 0$. Substitute the above $\mathcal{O}^{(1)}(k)[M_i]$ into (21), $\mathcal{O}(k)[M_i^p] = -piM_i^{p-1}M_{i-1}L - pM_i^{p-1}\mathcal{O}^{(2)}[M_i] + \mathcal{O}^{(2)}[M_i^p]$. Note that $\frac{\partial M_i^p}{\partial x} = pM_i^{p-1}\frac{\partial M_i}{\partial x}$, and $\frac{\partial^2 M_i^p}{\partial x^2} = p(p-1)M_i^{p-2}(\frac{\partial M_i}{\partial x})'(\frac{\partial M_i}{\partial x}) + pM_i^{p-1}\frac{\partial^2 M_i}{\partial x^2}$. Then we have the following.

$$\begin{aligned}\mathcal{O}(k)[M_i^p] = {} & -piM_i^{p-1}M_{i-1}L - pM_i^{p-1}\mathcal{O}^{(2)}[M_i] \\ & + \frac{1}{2}tr\left(\sigma W\sigma' p(p-1)M_i^{p-2}\left(\frac{\partial M_i}{\partial x}\right)\left(\frac{\partial M_i}{\partial x}\right)'\right) \\ & + pM_i^{p-1}\mathcal{O}^{(2)}[M_i],\end{aligned}$$

which gives the desired result. □

Lemma 4.3. *The powers of the cost moments M_iM_j are related by the following partial differential equation,*

$$\begin{aligned}\mathcal{O}(k)[M_i^pM_j^q] = {} & -piM_i^{p-1}M_j^qM_{i-1}L - qjM_i^pM_j^{q-1}M_{j-1}L \\ & + \frac{p(p-1)}{2}M_i^{p-2}M_j^q\left\|\frac{\partial M_i}{\partial x}\right\|_{\sigma W\sigma'} \\ & + qpM_i^{p-1}M_j^{q-1}tr \\ & \times\left(\sigma W\sigma'\left(\frac{\partial M_j}{\partial x}\right)\left(\frac{\partial M_i}{\partial x}\right)'\right) \\ & + \frac{q(q-1)}{2}M_i^pM_j^{q-2}\left\|\frac{\partial M_j}{\partial x}\right\|^2_{\sigma W\sigma'}.\end{aligned}$$

Proof. This is proved in the similar manner as Lemma 4.2. We omit the proof for brevity. □

Remarks (*The Procedure*)**.** Now we propose a procedure to find the nth order cost cumulant HJB equation: (a) use the moment–cumulant relationship to find the relationship between the nth moment and nth cumulant, see Stuart and Ord (1987, p. 85) and Won (2005); (b) substitute M_n into (6) and use Lemmas 4.1–4.3 to find the nth cumulant HJB equation.

Using this procedure, it is possible to determine any nth cost cumulant HJB equation. As examples, we will find the second, third and fourth cumulant HJB equations. Even though the second cumulant case is completely developed in Sain et al. (2000), we note that we use a different method to derive the following necessary condition. The importance of the following derivation is that this procedure allows an extension to higher order cumulants.

Now, we present the HJB equation for the second cost cumulant. If an optimal controller exists, then it has to satisfy the derived HJB equation. This is a necessary condition for optimality.

Theorem 4.1. *Let $M_1 \in C_p^{1,2}(\bar{Q}_0)$ be an admissible mean cost function, and let M_1 induce a nonempty class K_{M_1} of admissible control laws. Assume the existence of an optimal control law $k = k^*_{V_2|M_1}$ and an optimum value function $V_2^* \in C_p^{1,2}(\bar{Q}_0)$. Then the minimal second cumulant (variance) function V_2^* satisfies the following HJB equation.*

$$\min_{k\in K_{M_1}} \mathcal{O}(k)[V_2^*(t,x)] + \left\|\frac{\partial V_1(t,x)}{\partial x}\right\|^2_{\sigma W\sigma'} = 0 \tag{22}$$

for $(t, x) \in \bar{Q}_0$, together with the terminal condition, $V_2^*(t_F, x) = 0$.

Proof. From the moment–cumulant relationship equation in Stuart and Ord (1987), we have $M_2 = V_2 + M_1^2$. Substitute this into (6) to obtain $\min_{k\in K_{M_1}}\{\mathcal{O}(k)[V_2^*(t,x) + M_1^2(t,x)] + 2M_1(t,x)L(t,x,k)\} = 0$. Thus, to obtain the second cumulant HJB equation, it is necessary and sufficient to show that

$$\mathcal{O}(k)[M_1^2(t,x)] + 2M_1(t,x)L(t,x,k(t,x)) = \left\|\frac{\partial M_1(t,x)}{\partial x}\right\|^2_{\sigma W\sigma'} \tag{23}$$

whenever $k \in K_{M_1}$. Using Lemma 4.3 with $p = 2$ and $M_1 = V_1$, we have the desired relationship. □

Remarks. If we do not constrain the controller to be in K_{M_1}, and seek the HJB equation for the second cumulant case, then we obtain a degenerate HJB equation of the form $\min_k \mathcal{O}(k)[V_2^*(t,x)] = 0$. Consequently, the sum of a first two cumulant problem would give the same optimal controller as the minimal mean case, if the controllers are not constrained.

The following theorem states that if a solution to the following HJB equation exists then it is an optimal cost and the controller is an optimal controller. This verifies that the controller is indeed optimal.

Theorem 4.2 (*Second Cumulant Verification Theorem*). *Let $M_1 \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be an admissible mean cost function. Let $V_2^* \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be a solution to the partial differential equation*

$$\min_{k \in K_{M_1}} \mathcal{O}(k)\left[V_2^*(t,x)\right] + \left\| \frac{\partial V_1(t,x)}{\partial x} \right\|^2_{\sigma W \sigma'} = 0, \qquad (24)$$

where $V_2^*(t_f, x) = 0$. Then $V_2^*(t,x) \le Var\{J(t,x,k(t,x))\}$ for all $k \in K_{M_1}$ and $(t,x) \in Q$. If in addition, such a k satisfies the following equation,

$$\mathcal{O}(k)[V_2^*(t,x)] = \min_{k \in K_{M_1}} \{\mathcal{O}(k)[V_2^*(t,x)]\},$$

then $V_2^*(t,x) = Var\{J(t,x,k)\}$ and $k = k^*_{V_2|M_1}$ is an optimal controller.

Proof. By definition of V_2^*, we have $V_2^*(t,x) = M_2^*(t,x) - M_1^2(t,x)$. Substituting into (24) yields

$$\min_{k \in K_{M_1}} \mathcal{O}(k)\left[M_2^*(t,x) - M_1^2(t,x)\right] + \left\| \frac{\partial V_1(t,x)}{\partial x} \right\|^2_{\sigma W \sigma'} = 0.$$

Notice that with the identity (23) in the previous theorem's proof and with some manipulation we obtain

$$\min_{k \in K_{M_1}} \mathcal{O}(k)\left[M_2^*(t,x)\right] + 2M_1(t,x)L(t,x,k(t,x)) = 0,$$

where this is the sufficient condition given in Theorem 3.2. □

We present the third cumulant HJB equation. The following theorem is a necessary condition for the optimal controller, which is obtained via a HJB equation. If an optimal controller exists then it will necessarily satisfy the following theorem.

Theorem 4.3. *Let $M_1 \in C_p^{1,2}(\bar{Q}_0)$ be an admissible mean cost function, and let M_1 induce a nonempty class K_{M_1} of admissible control laws. Assume the existence of an optimal control law $k = k^*_{V_3|M_1}$ and an optimum value function $V_3^* \in C_p^{1,2}(\bar{Q}_0)$. Then the minimal third cost cumulant (skewness) function V_3^* satisfies the following HJB equation.*

$$\min_{k \in K_{M_1}} \left\{ \mathcal{O}(k)[V_3^*(t,x)] + 3tr\left(\sigma W \sigma' \left(\frac{\partial V_1(t,x)}{\partial x}\right)\left(\frac{\partial V_2(t,x)}{\partial x}\right)'\right)\right\} = 0 \qquad (25)$$

for $(t,x) \in \bar{Q}_0$, together with the terminal condition, $V_3^*(t_F, x) = 0$.

Proof. From the cumulant–moment relationship given in Stuart and Ord (1987, p. 85) and Won (2005), we have

$$V_3 = M_3 - 3M_2M_1 + 2M_1^3.$$

Substitute this into (6) to obtain

$$\min_{k \in K_{M_1}} \left\{\mathcal{O}(k)[V_3^*(t,x) + 3M_2M_1 - 2M_1^{*3}] + 3M_2L(t,x,k)\right\} = 0.$$

For brevity we will suppress the arguments. Now, we need to show that

$$\mathcal{O}(k)[3M_2M_1 - 2M_1^3] + 3M_2L = 3tr\left(\sigma W \sigma' \left(\frac{\partial V_1}{\partial x}\right)\left(\frac{\partial V_2}{\partial x}\right)'\right), \qquad (26)$$

whenever $k \in K_{M_1}$. Using Lemma 4.1, we have

$$\mathcal{O}(k)[M_1^3] = -3M_1^2L + 3M_1 \left\| \frac{\partial M_1}{\partial x} \right\|^2_{\sigma W \sigma'}. \qquad (27)$$

Using Lemma 4.2, we have

$$\mathcal{O}(k)[M_2M_1] = -2M_1^2L - M_2L + tr\left(\sigma W \sigma' \left(\frac{\partial M_1(t,x)}{\partial x}\right)\left(\frac{\partial M_2(t,x)}{\partial x}\right)'\right). \qquad (28)$$

Using the moment to cumulant relationship in Won (2005), we have $M_1 = V_1$, and $M_2 = V_2 + V_1^2$. Therefore,

$$\frac{\partial M_1}{\partial x} = \frac{\partial V_1}{\partial x} \qquad (29)$$

$$\frac{\partial M_2}{\partial x} = \frac{\partial V_2}{\partial x} + \frac{\partial V_1^2}{\partial x} = \frac{\partial V_2}{\partial x} + 2V_1\frac{\partial V_1}{\partial x}. \qquad (30)$$

Utilizing (27) and (28), we have

$$\begin{aligned}
&\mathcal{O}(k)[3M_2M_1 - 2M_1^3] + 3M_2L = -6M_1^2L - 3M_2L \\
&\quad + 3tr\left(\sigma W \sigma' \left(\frac{\partial M_1(t,x)}{\partial x}\right)\left(\frac{\partial M_2(t,x)}{\partial x}\right)'\right) \\
&\quad + 6M_1^2L - 6M_1 \left\| \frac{\partial M_1}{\partial x} \right\|^2_{\sigma W \sigma'} + 3M_2L \\
&= 3tr\left(\sigma W \sigma' \left(\frac{\partial M_1(t,x)}{\partial x}\right)\left(\frac{\partial M_2(t,x)}{\partial x}\right)'\right) \\
&\quad - 6M_1 tr\left(\sigma W \sigma' \left(\frac{\partial M_1(t,x)}{\partial x}\right)\left(\frac{\partial M_1(t,x)}{\partial x}\right)'\right).
\end{aligned}$$

Now we use (29) and (30). We obtain

$$\begin{aligned}
&\mathcal{O}(k)[3M_2M_1 - 2M_1^3] + 3M_2L \\
&= 3tr\left(\sigma W \sigma' \left(\frac{\partial V_1(t,x)}{\partial x}\right)\left(\frac{\partial V_2(t,x)}{\partial x} + 2V_1\frac{\partial V_1}{\partial x}\right)'\right) \\
&\quad - 6V_1 tr\left(\sigma W \sigma' \left(\frac{\partial V_1(t,x)}{\partial x}\right)\left(\frac{\partial V_1(t,x)}{\partial x}\right)'\right) \\
&= 3tr\left(\sigma W \sigma' \left(\frac{\partial V_1(t,x)}{\partial x}\right)\left(\frac{\partial V_2(t,x)}{\partial x}\right)'\right).
\end{aligned}$$

We proved (26), and which in turn proves (25). □

Remarks. We note that we find the admissible control law, k, from K_{M_1} and that k gives a corresponding $V_1(t,x)$ and $V_2(t,x)$. Thus, we are not optimizing with respect to the first and second cumulant in this case.

The following theorem is a sufficient condition for the optimal controller.

Theorem 4.4 (*Third Cumulant Verification Theorem*). *Let $M_1 \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be an admissible mean cost function. Let $V_3^* \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be a solution to the partial differential equation*

$$\min_{k \in K_{M_1}} \mathcal{O}(k)\left[V_3^*(t,x)\right] + 3tr\left(\sigma W \sigma' \left(\frac{\partial V_1(t,x)}{\partial x}\right)\left(\frac{\partial V_2(t,x)}{\partial x}\right)'\right) = 0. \qquad (31)$$

Then $V_3^*(t,x)$ is less than or equal to the third cumulant of the cost $J(t,x,k(t,x))$ for all $k \in K_{M_1}$ and $(t,x) \in Q$. If in addition, such a k satisfies the following equation,

$$\mathcal{O}(k)[V_3^*(t,x)] = \min_{k \in K_{M_1}} \{\mathcal{O}(k)[V_3^*(t,x)]\},$$

then $V_3^*(t,x)$ equals the third cumulant of $J(t,x,k)$ and $k = k^*_{V_3|M_1}$ is an optimal controller.

Proof. By definition of V_3^*, we have $V_3^*(t,x) = M_3^*(t,x) - 3M_2(t,x)M_1(t,x) + 2M_1^3(t,x)$. Substituting into (31) yields

$$\min_{k\in K_{M_1}} \mathcal{O}(k)\left[M_3^*(t,x) - 3M_2(t,x)M_1(t,x) + 2M_1^3(t,x)\right] + 3tr\left(\sigma W\sigma'\left(\frac{\partial V_1(t,x)}{\partial x}\right)\left(\frac{\partial V_2(t,x)}{\partial x}\right)'\right) = 0.$$

Notice that with the identity (26) in the previous theorem's proof, and with some manipulation, we obtain

$$\min_{k\in K_{M_1}} \mathcal{O}(k)\left[M_3^*(t,x)\right] + 3M_2(t,x)L(t,x,k(t,x)) = 0,$$

where this is the sufficient condition given in Theorem 3.2. □

The minimization of a second cumulant (variance) depends on the definition of the first cumulant (mean) value function V_1 as can be seen from (22). The minimization of a third cumulant (skewness) depends on the definition of both the first and second cumulant value functions V_1 and V_2 as can be seen from (25). Here, we present the fourth cumulant (kurtosis) HJB equation as a necessary condition for optimality.

Theorem 4.5. *Let $M_1 \in C_p^{1,2}(\bar{Q}_0)$ be an admissible mean cost function, and let M_1 induce a nonempty class K_{M_1} of admissible control laws. Assume the existence of an optimal control law $k = k^*_{V_4|M_1}$ and an optimum value function $V_4^* \in C_p^{1,2}(\bar{Q}_0)$. Then the minimal fourth cost cumulant (kurtosis) function V_4^* satisfies the following HJB equation,*

$$\min_{k\in K_{M_1}} \mathcal{O}(k)[V_4^*(t,x)] + 3\left\|\frac{\partial V_2}{\partial x}\right\|^2_{\sigma W\sigma'} + 4tr\left(\sigma W\sigma'\left(\frac{\partial V_1(t,x)}{\partial x}\right)\left(\frac{\partial V_3(t,x)}{\partial x}\right)'\right) = 0, \quad (32)$$

for $(t,x) \in \bar{Q}_0$, together with the terminal condition, $V_4^*(t_F,x) = 0$.

Proof. The proof is similar to the previous theorem, and is omitted for brevity. □

The following sufficient condition for optimality is presented in the form of a verification theorem.

Theorem 4.6 (*Fourth Cumulant Verification Theorem*)**.** *Let $M_1 \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be an admissible mean cost function. Let $V_4^* \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be a solution to the partial differential equation*

$$\min_{k\in K_{M_1}} \mathcal{O}(k)\left[V_4^*(t,x)\right] + \left\|\frac{\partial V_2(t,x)}{\partial x}\right\|^2_{\sigma W\sigma'} + 3tr\left(\sigma W\sigma'\left(\frac{\partial V_1(t,x)}{\partial x}\right)\left(\frac{\partial V_2(t,x)}{\partial x}\right)'\right) = 0. \quad (33)$$

Then $V_4^*(t,x)$ is less than or equal to the fourth cumulant of the cost $J(t,x,k(t,x))$ for all $k \in K_{M_1}$ and $(t,x) \in Q$.

If in addition, such a k satisfies the following equation,

$$\mathcal{O}(k)[V_4^*(t,x)] = \min_{k\in K_{M_1}}\{\mathcal{O}(k)[V_4^*(t,x)]\},$$

then $V_4^*(t,x)$ equals the fourth cumulant of $J(t,x,k)$ and $k = k^*_{V_4|M_1}$ is an optimal controller.

Proof. This proof is similar to the previous proofs for the second and third cumulant verification theorems and is omitted for brevity. □

Remarks. The presented theorems in this paper allow the derivations of first to fifth order cost cumulant HJB equations. We provide necessary and sufficient conditions for second, third, and fourth cost cumulant statistical control. In order to determine the higher order statistical control HJB equations, we require more partial differential equation lemmas. The general nth order case is left as the future work. We note however, the cumulants have decreasing significance as the order of cumulants increases, so having up to a fourth order cost cumulant control is a significant result.

Remarks. In this paper, the shaping of the cost density is done by constraining the admissible control law to have a certain mean. So, we can reduce the cost variance (measures the degree of being spread out), cost skewness (measures of asymmetry), and cost kurtosis (measures peakedness) within the admissible controllers. If we optimize the sum of the first two cumulants without the mean constraint, we obtain the controller that is the same as the minimal mean controller. This is because of the Gaussian assumption on the disturbance function. If a different distribution is used for the disturbance function, this will not be true.

5. Second cost cumulant optimization solution of a control-affine nonlinear system

We solve for the full-state-feedback solution of the optimal statistical control problem numerically for the first and second cost cumulant case when the system is nonlinear in state. The problem formulation starts from (1) and (2). We make a few assumptions: $L(t,x,k(t,x)) = l(t,x) + k'(t,x)R(t,x)k(t,x)$, $\psi(x(t_F)) = 0$, and $f(t,x,k(t,x)) = g(t,x) + B(t,x)k(t,x)$, where k is an admissible (*i.e.*, satisfies local Lipschitz condition and linear growth condition) feedback control law; $l : \bar{Q}_0 \to \mathbb{R}^+$ is $C(\bar{Q}_0)$ and satisfies the polynomial growth conditions assumed for L; and $g : \bar{Q}_0 \to \mathbb{R}^n$ is $C^1(\bar{Q}_0)$ and satisfies the linear growth condition and the local Lipschitz condition assumed for f. Moreover $R(t,x) > 0$, and $B(t,x)$ are continuous real matrices of appropriate dimensions for all $t \in T$. The state equation that we are considering is

$$dx(t) = g(t,x)dt + B(t,x)k(t,x)dt + \sigma(t,x)dw(t) \quad (34)$$

and the cost function is

$$J(t,x,k) = \int_t^{t_F} l(t,x) + k'(t,x)R(t,x)k(t,x)dt \quad (35)$$

where $E\{dw(t)dw'(t)\} = W(t)dt$.

Now, the problem is formulated. We presuppose that a cost mean, $V_1(t,x) = E_{tx}\{J(t,x,k)\}$, not necessarily minimal, has been specified, then we seek to minimize the cost variance, $V_2(t,x)$.

From (19) and (22), we have the following two partial differential equations as the necessary conditions for optimality.

$$\frac{\partial V_1}{\partial t} + g'\frac{\partial V_1}{\partial x} + k'B'\frac{\partial V_1}{\partial x} + \frac{1}{2}tr\left(\sigma W\sigma'\frac{\partial^2 V_1}{\partial x^2}\right) + l + k'Rk = 0, \quad (36)$$

$$\frac{\partial V_2^*}{\partial t} + g'\frac{\partial V_2^*}{\partial x} + k'B'\frac{\partial V_2^*}{\partial x} + \frac{1}{2}tr\left(\sigma W\sigma'\frac{\partial^2 V_2^*}{\partial x^2}\right) + tr\left(\sigma W\sigma'\frac{\partial V_1}{\partial x}\left(\frac{\partial V_1}{\partial x}\right)'\right) = 0. \quad (37)$$

We use the Lagrange multiplier method. Introduce a time varying Lagrange multiplier, $\gamma_2(t)$, and optimize the HJB equation for the pre-specified first cumulant, $V_1(t,x)$, plus the Lagrange multiplier times the HJB equation of the second cumulant, $V_2(t,x)$. It is this second cumulant that we minimize within the admissible controllers.

$$
\begin{aligned}
0 = \min_{k} \Bigg\{ & \frac{\partial V_1}{\partial t} + g'\frac{\partial V_1}{\partial x} + k'B'\frac{\partial V_1}{\partial x} + \frac{1}{2} tr\left(\sigma W \sigma' \frac{\partial^2 V_1}{\partial x^2}\right) \\
& + l + k'Rk + \gamma_2 \left(\frac{\partial V_2^*}{\partial t} + g'\frac{\partial V_2^*}{\partial x} + k'B'\frac{\partial V_2^*}{\partial x}\right. \\
& \left. + \frac{1}{2} tr\left(\sigma W \sigma' \frac{\partial^2 V_2^*}{\partial x^2}\right) + tr\left(\sigma W \sigma' \frac{\partial V_1}{\partial x}\left(\frac{\partial V_1}{\partial x}\right)'\right)\right)\Bigg\}. \qquad (38)
\end{aligned}
$$

The minimizing controller is obtained as

$$
\begin{aligned}
k^*(t,x) = & -\frac{1}{2} R^{-1}(t,x) B'(t,x) \\
& \times \left[\frac{\partial V_1(t,x)}{\partial x} + \gamma_2(t)\frac{\partial V_2^*(t,x)}{\partial x}\right]. \qquad (39)
\end{aligned}
$$

The second order necessary condition, $R(t,x) > 0$, is satisfied. Therefore, the minimum is guaranteed, and the controller (39) is a candidate for an optimal statistical controller for the first two cost cumulant case.

Remark. We obtain the minimal mean (first cumulant) case when we let $\gamma_2 = 0$ in (39).

Remarks. We note that the cost moment control are more complicated than cost cumulant control. If we were to perform the second cost moment minimization, we will obtain a nonlinear controller even for the linear, $g = Ax$, quadratic cost case, $L(t,x,k) = x'Qx + k'Rk$. To see this, we find the admissible controller that gives M_1, and we minimize the second moment. Consider the second moment HJB equation from (5):

$$
\begin{aligned}
0 = & \frac{\partial M_2}{\partial t} + 2M_1(x'Qx + k'Rk) + \frac{1}{2} tr\left(\sigma W \sigma' \frac{\partial^2 M_2}{\partial x^2}\right) \\
& + x'A'\frac{\partial M_2}{\partial x} + k'B'\frac{\partial M_2}{\partial x}.
\end{aligned}
$$

We find the optimal second moment controller as $k^* = -\frac{1}{2+4\gamma M_1} R^{-1}B'\left(\frac{\partial M_1}{\partial x} + \gamma \frac{\partial M_2}{\partial x}\right)$. If we let $M_i(t,x) = x'\mathscr{M}_i x + m_i$ for $i = 1, 2$, we obtain a nonlinear controller.

There are a number of approaches that one can take to obtain the solutions of the HJB equations: find solutions of the HJB equation using the state dependent Riccati equation (SDRE) method (Cloutier, D'Souza, & Mracek, 1996; Won, 2005), pseudo-inversion method, and numerical method (Beard, 0000; Cheng, Lewis, & Abu-Khalaf, 2007; Navasca & Krener, 2000). However, in general the analytical solutions for HJB equations (36) and (37) are difficult to obtain except for low order systems. Therefore, here we will find the solution of the second cumulant case numerically.

Inspired by the Galerkin's method, which numerically approximates HJB equation (Beard, 0000), and the neural network method proposed in Cheng et al. (2007), we apply a numerical neural network method combined with a weighted residual method to find the approximate solutions for HJB equations (36) and (37).

We use $V_{1L}(x,t) = w_L'(t)\delta_L(x)$ to approximate V_1, and $V_{2L}(x,t) = v_L'(t)\delta_L(x)$ to approximate V_2 in Eqs. (36) and (37) respectively, where $w_L(t)$ and $v_L(t)$ are vectors which contain the time varying weights of the neural network function set $\delta_L(x)$. And the neural network functions in the set $\delta_L(x)$ are chosen to be linearly independent in domain Ω. Usually, we choose $\delta_L(x)$ as a set of linearly independent polynomial functions. The domain Ω is chosen such that an admissible control k exists in Ω. The letter L denotes the number of the neural network functions which determines the accuracy of the solution for the HJB equations. More neural network functions are used in the calculation, the accuracy of the solution improves.

Applying the weighted residual method, we convert Eqs. (36) and (37) to ordinary differential equations with respect to the unknowns $w_L(t)$ and $v_L(t)$, as shown in Eqs. (40) and (41). We use the following notations in the equations. $A = \langle \delta_L(x), \delta_L(x)\rangle_\Omega^{-1}$, $B = \langle \nabla\delta_L(x) g(x), \delta_L(x)\rangle_\Omega$, $C_i = \langle \nabla\delta_L(x) B(t,x) R^{-1} B'(t,x) \frac{\partial \delta_i(x)}{\partial x}, \delta_L(x)\rangle_\Omega$, $D = \langle l, \delta_L(x)\rangle_\Omega$,$E = \langle tr(\sigma W \sigma' \frac{\partial(\nabla\delta_L'(x) w_L(t))}{\partial x}), \delta_L(x)\rangle_\Omega$, and $F_i = \langle \nabla\delta_L(x)\sigma W \sigma' \frac{\partial \delta_i(x)}{\partial x}, \delta_L(x)\rangle_\Omega$, then, we have

$$
\begin{aligned}
\dot{w}_L(t) = & -A^{-1}Bw_L(t) + \frac{1}{4}A^{-1}\left(\sum_{i=1}^{L} w_i C_i\right) w_L(t) \\
& - \frac{\gamma_2^2}{4} A^{-1}\left(\sum_{i=1}^{L} v_i C_i\right) v_L(t) - A^{-1}D - \frac{1}{2}A^{-1}E, \qquad (40)
\end{aligned}
$$

$$
\begin{aligned}
\dot{v}_L(t) = & -A^{-1}Bv_L(t) + \frac{1}{2}A^{-1}\left(\sum_{i=1}^{L} w_i C_i\right) v_L(t) - \frac{1}{2}A^{-1}E \\
& + \frac{\gamma_2}{2}A^{-1}\left(\sum_{i=1}^{L} v_i C_i\right) v_L(t) - A^{-1}\left(\sum_{i=1}^{L} w_i F_i\right) w_L(t) \qquad (41)
\end{aligned}
$$

and the optimal controller is given by

$$
\begin{aligned}
k^* & = -\frac{1}{2}R^{-1}B'\left[\frac{\partial V_{1L}(t,x)}{\partial x} + \gamma_2(t)\frac{\partial V_{2L}^*(t,x)}{\partial x}\right] \\
& = -\frac{1}{2}R^{-1}B'\left[w_L'(t)\frac{\mathrm{d}\delta_L(x)}{\mathrm{d}x} + \gamma_2(t) v_L'(t)\frac{\mathrm{d}\delta_L(x)}{\mathrm{d}x}\right], \qquad (42)
\end{aligned}
$$

where the partial derivatives of V_{1L} and V_{2L} are changed into full derivatives of $\delta_L(x)$, because $\delta_L(x)$ is not explicitly dependent to t.

Here $\langle\cdot,\cdot\rangle_\Omega$ denotes the inner product over the domain Ω. Eqs. (40) and (41) can be solved numerically for $w_L(t)$ and $v_L(t)$ with the given terminal conditions. Therefore, we find $V_{1L}(x,t)$ and $V_{2L}(x,t)$. The optimal controller k^* is found via Eq. (42). We will illustrate this method through an example.

Example. As an example, consider the following stochastic system $\mathrm{d}x(t) = [\sin x(t)]\mathrm{d}t + u(t)\mathrm{d}t + \mathrm{d}w(t)$ with a cost function $J = \int_{t_0}^{t_F} 0.5(x^2 + u^2)\mathrm{d}t$.

Both the Lipschitz condition and the growth condition are satisfied in this example. Now, we find the optimal controller by solving the minimal first and second cumulant control problem, which is to find $V_2^*(x)$ from an admissible controller, k, in K_{M1}. In this example, we take $g(x) = \sin x$, $B = 1$, $P = BR^{-1}B' = 2$, $l(x) = x^2/2$, and $\sigma = 1$. The covariance matrix of $\mathrm{d}w$ is W.

To find the optimal controller, we must solve the HJB equations (36) and (37), therefore, we use the neural network method described above. In this example, we choose the neural network function set $\delta_L(x) = \{1, x^2, x^4, x^6, x^8, x^{10}, x^{12}, x^{14}, x^{16}\}$, here the number of the neural network functions, L, is nine. Then, we apply the neural network method, and substitute $V_{1L}(x,t) = w_L'(t)\delta_L(x)$ and $V_{2L}(x,t) = v_L'(t)\delta_L(x)$ to the Eqs. (40) and (41). The $V_{1L}(x,t)$ and $V_{2L}(x,t)$ are found numerically. If we assume that $x = 1$, $\gamma_2 = 1$, and $W = 1$, then from Eq. (42), the optimal controller is $k^* = -2.0582$.

6. Conclusions

This paper presented a method to control the distribution of a nonquadratic cost function for a control-affine nonlinear system by controlling the cost cumulants. Statistical control is used to shape the cost distribution using higher order cumulants. We derived the necessary condition for the nth cost moment optimization problem and use this result to derive higher order cumulant HJB equations. We also presented the sufficient condition for optimality in the form of a verification theorem. We derived second, third, and fourth cost cumulant HJB equations as the necessary conditions for optimality. We minimize the higher order cumulants among the admissible controllers. This method can be used to derive fifth, sixth, and higher order HJB equations. The verification theorems are also presented for second, third, and fourth cost cumulants. These are the sufficient conditions for the optimal controllers. The solution procedure for the cost cumulant problem is also given in this paper. We used a neural network approach to solve the HJB equations. By using the neural network method, the HJB equations are converted to the first order ordinary differential equations and are solved numerically. We showed that the neural network method numerically solves the statistical control problem of the second cost cumulant minimization case for the control-affine nonlinear system.

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