

# Bi-Cumulant Games: A Generalization of $H_\infty$ and $H_2/H_\infty$ Control

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## Abstract

Cost cumulants have been used more frequently in the recent literature and have demonstrated their usefulness in controlling the vibrations of buildings that are subject to high winds and seismic disturbances. This paper extends the use of cumulants into stochastic game theory. The development is for a class of nonlinear systems with non-quadratic costs, over a finite time horizon. Sufficient and necessary conditions for Nash and minimax equilibrium solutions are given; and, for the case of linear systems with quadratic costs, the equilibrium solutions are determined. In the process, coupled Riccati equations are found. It is shown that these bi-cumulant games constitute a generalization of  $H_\infty$  and  $H_2/H_\infty$ . Moreover, the  $H_\infty$  generalization is applied to the control of a four-story building under a seismic disturbance; and the results are compared with those of linear quadratic gaussian, minimum cost variance, and  $H_\infty$  control.

## I. INTRODUCTION

Stochastic optimal control has historically been based in great part upon the mean of a cost function. The mean of the cost as used in Linear Quadratic Gaussian (LQG) control is a useful measure of performance (see [5]); but one

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may also wonder what kind of performance can be obtained by using other statistical quantities, such as the cost variance. This is the type of question that cumulant control addresses. In cumulant control, a performance index is minimized that is based upon cumulants. Cumulants are a complete set of statistics that include the mean, variance, and skewness. They are found from the second characteristic function, which is simply the natural logarithm of the first characteristic function. The work was started with the minimum variance control problem by Sain in [29], [30], [31], [32]. The discrete time version of this problem was then studied by Cosenza in [7]. The quadratic nature of cumulants was examined by Liberty in [19], as well as the associated cost densities, in this work and in [34]. Extensions of cumulants beyond the cost mean and variance were found by Liberty and Hartwig in [21]. The work of Won [38]; Sain, Won, and Spencer [36]; and Sain, Won, Spencer, Liberty [37] examined minimum cost variance control (MVC) in a nonlinear system and non-quadratic cost framework, in which necessary and sufficient conditions for the MVC control laws were found. In the work of Pham, *et al.* [23]- [26], the finite horizon  $k$ -cumulant state feedback control method was developed. This work let the control minimize a linear combination of cumulants of a quadratic cost function for a linear system. The infinite time horizon results were then given in Won, *et al.* [39] and Pham, *et al.* [27], and the output feedback results in [28]. Furthermore, these results and the infinite time horizon, output feedback problem are given in Pham's dissertation, [26]. The application of cumulants to stochastic control has positive results. Both minimum cost variance (MCV) and  $k$  cost cumulant ( $k$ CC) control have been applied to structural vibration problems with encouraging outcomes.

Because cost cumulants include the mean, one can view this vein of research as a generalization of LQG control, which was popularized by R. E. Kalman. It should be noted that this method of control has one performance index which involves multiple cumulants. In a similar manner to stochastic control, in stochastic game theory the mean of the cost plays a large role. Basar [3] extended the use of the mean in stochastic differential games with a risk-sensitive Nash game, and for the discrete-time, Klompstra [18], did the same. Stochastic games can be extended in a different direction with the use of cumulants. In this problem each player's performance index can be based upon the mean and higher order cumulants.

While the problem of extending game theory to include the use of more cost cumulants is a worthy task, more control related aspects of this extension are generalizations of  $H_2/H_\infty$  and  $H_\infty$  control. In Limebeer *et al* [22], it was found that the  $H_2/H_\infty$  control problem can be solved with a two player game. The two players were the control

$u$  and disturbance  $w$ . For the deterministic case, the two players were interested in determining their equilibrium solution based upon the cost functions

$$J_1 = \int_{t_0}^{t_f} z'(t)z(t)dt$$

$$J_2 = \int_{t_0}^{t_f} [\delta^2 w'(t)w(t) - z'(t)z(t)] dt,$$

where  $z(t) = C(t)x(t) + D(t)u(t)$ ,  $i = 1, 2$ , is a regulated output of a system and  $u, w$  wish to minimize  $J_1$  and  $J_2$ , respectively. With this form of the game problem, a Nash equilibrium solution was sought. What was found in [22] was that, by the disturbance wishing to minimize  $J_2$ , the result was a constraint on the  $H_\infty$  norm of the system from  $w$  to  $z$ ; and in fact this constraint is the constant  $\delta$ . In the stochastic version of this problem the players then wish to minimize the mean of their cost functions. Similarly in [2], the  $H_\infty$  control problem is developed through game theory, except that in this case, instead of a Nash game, a minimax game is discussed. Here there is one cost function where the disturbance  $w$  wants to maximize, while the control  $u$  wants to be the minimizer. Also, for the stochastic case, the players are concerned with the mean of the cost. So if the  $H_2/H_\infty$  and  $H_\infty$  control problems have stochastic versions in which the mean of the cost is examined, then we can study the advantages to be gained by broadening these results with the use of higher order cumulants. By so extending the performance index to higher order cumulants, we generalize the well known  $H_2/H_\infty$  and  $H_\infty$  control problems

In this paper, cumulants will be used to extend stochastic differential game theory. Two games will be discussed: a Nash game and a minimax game, in which the players will be concerned also with the variance of their costs. The game will be developed, in both cases, for a class of nonlinear systems with non-quadratic costs. In the next section, preliminary assumptions on the system will be discussed. Then, for the non-zero sum game, the problem is defined and sufficient and necessary conditions are given for the players' equilibrium solutions. The zero sum case is discussed in the same manner, following the work of non-zero sum game. With the conditions given for the nonlinear system with non-quadratic costs, equilibrium solutions are found for the linear quadratic case. Finally, the generalization of  $H_2/H_\infty$  and  $H_\infty$  is discussed, and the results are applied to a four story building under a seismic disturbance. An earlier version of this work was given in [12].

## II. PRELIMINARIES

Consider the following stochastic differential equation

$$dx(t) = f(t, x(t), u(t), w(t))dt + \sigma(t, x(t))d\xi(t) \quad (1)$$

where  $x(t_0) = x_0$  is a random variable independent of  $\xi$ ,  $\xi$  is  $d$ -dimensional Brownian motion on the probability space  $(\Omega, \mathcal{F}, P)$ ,  $x(t) \in \mathbb{R}^n$  is the state,  $u(t) \in \mathcal{U}$  is the control,  $w(t) \in \mathcal{W}$  is the disturbance, and  $t \in T = [t_0, t_f]$ . Let  $Q_0 = (t_0, t_f) \times \mathbb{R}^n$  and  $\bar{Q}_0$  be its closure, that is  $\bar{Q}_0 = T \times \mathbb{R}^n$ . Assume the functions  $f$  and  $\sigma$  to be Borel measurable and of class  $C^1(\bar{Q}_0 \times \mathcal{U} \times \mathcal{W})$  and  $C^1(\bar{Q}_0)$  respectively. This means that the functions  $f$  and  $\sigma$  have continuous partial derivatives of first order. Furthermore assume that  $f$  and  $\sigma$  satisfy the following conditions:

(i) there exists a constant  $C$  such that

$$\|f(t, x, u, w)\| \leq C(1 + \|x\| + \|u\| + \|w\|)$$

$$\|\sigma(t, x)\| \leq C(1 + \|x\|);$$

for all  $(t, x, u, w) \in \bar{Q}_0 \times \mathcal{U} \times \mathcal{W}$ ,  $(t, x) \in \bar{Q}_0$ , and  $\|\cdot\|$  is the Euclidean norm.

(ii) and there is a constant  $K$  so that

$$\begin{aligned} \|f(t, \tilde{x}, \tilde{u}, \tilde{w}) - f(t, x, u, w)\| &\leq K(\|\tilde{x} - x\| + \|\tilde{u} - u\| \\ &\quad + \|\tilde{w} - w\|) \end{aligned}$$

$$\|\sigma(t, \tilde{x}) - \sigma(t, x)\| \leq K\|\tilde{x} - x\|$$

for all  $t \in T$ ;  $x, \tilde{x} \in \mathbb{R}^n$ ;  $u, \tilde{u} \in \mathcal{U}$ ;  $w, \tilde{w} \in \mathcal{W}$ .

We also assume some conditions on the strategies of the control and disturbance. First assume that the strategies are of the form  $u(t) = \mu(t, x(t))$  and  $w(t) = v(t, x(t))$ . Furthermore the functions  $\mu : \bar{Q}_0 \rightarrow \mathcal{U}$  and  $v : \bar{Q}_0 \rightarrow \mathcal{W}$  are assumed to be Borel measurable and to satisfy:

(i) for some constant  $\tilde{C}$

$$\|\mu(t, x)\| \leq \tilde{C}(1 + \|x\|) \quad \text{and} \quad \|v(t, x)\| \leq \tilde{C}(1 + \|x\|),$$

(ii) and there exists a constant  $\tilde{K}$  such that

$$\|\mu(t, \tilde{x}) - \mu(t, x)\| \leq \tilde{K}(\|\tilde{x} - x\|)$$

$$\|v(t, \tilde{x}) - v(t, x)\| \leq \tilde{K}(\|\tilde{x} - x\|),$$

where  $t \in T$  and  $x, \tilde{x} \in \mathbb{R}^n$ . Often we will suppress the dependence on  $t$  and  $x$  and refer to the strategies as  $\mu$  and  $\nu$ . If the strategies  $\mu$  and  $\nu$  satisfy these conditions, then they are admissible strategies. We can rewrite the stochastic differential equation as

$$dx(t) = \tilde{f}(t, x(t))dt + \sigma(t, x(t))d\xi(t) \quad x(t_0) = x_0 \quad (2)$$

where the strategy  $(\mu, \nu)$  has been substituted into  $f$ , which is then denoted  $\tilde{f}$ . The conditions of Theorem V4.1 of [15] are now satisfied and we see that if  $E\|x(t_0)\|^2 < \infty$ , then a solution of (1) exists. Furthermore the solution  $x(t)$  is unique in the sense that if there exists another solution  $\tilde{x}(t)$  with  $\tilde{x}(t_0) = x_0$ , then the two solutions have the same sample paths with probability 1. The resulting process is a Markov diffusion process ([15] pg. 123) and the moments of  $x(t)$  are bounded.

Let  $C^{1,2}(\bar{Q}_0)$  be the class of functions  $\Phi$  that have continuous first partial derivatives with respect to  $t$  and continuous second partial derivatives with respect to  $x$ :  $\Phi_t, \Phi_{x_i}, \Phi_{x_i x_j}$  for  $i, j = 1, 2, \dots, n$ . Now let  $C_p^{1,2}(\bar{Q}_0)$  be the class of functions  $\Phi(t, x)$  that are of class  $C^{1,2}(\bar{Q}_0)$  but where  $\Phi, \Phi_t, \Phi_{x_i}, \Phi_{x_i x_j}$  satisfy a polynomial growth condition. A polynomial growth condition for a function  $\Phi$  is such that there exist constants  $c_1$  and  $c_2$  so that  $\|\Phi(t, x)\| \leq c_1(1 + \|x\|^{c_2})$  for all  $(t, x) \in \bar{Q}_0$ . This yields the Dynkin formula

$$\Phi(t, x) = E \left\{ \int_t^{t_f} -\mathcal{O}^{\mu, \nu} \Phi(s, x(s)) ds | x(t) = x \right\} + E \{ \Phi(t_f, x(t_f)) | x(t) = x \} \quad (3)$$

where  $\mathcal{O}^{\mu, \nu}$  is the backward evolution operator given by

$$\mathcal{O}^{\mu, \nu} = \frac{\partial}{\partial t} + f'(t, x, u, w) \frac{\partial}{\partial x} + \frac{1}{2} tr \left( \sigma(t, x) W(t) \sigma'(t, x) \frac{\partial^2}{\partial x^2} \right) \quad (4)$$

with  $E\{d\xi(t)d\xi'(t)\} = W(t)$ . The expectation in (3) will now be denoted by  $E_{tx}$ .

So far, the discussion has mainly been a preliminary one, in which the system has been given with two players, the control  $u$  and disturbance  $w$ . These two players may have one common cost function, or two separate cost functions, with which they are concerned.

### III. NON-ZERO SUM (NASH) GAME

#### A. Problem Definition

In non-zero sum, two player games, there is a performance index for each player; and they both wish to minimize their index when the other player has played their equilibrium solution. This sort of solution is called a Nash optimal

strategy. The game described by (1) has two cost functions. The first cost function,  $J_1$ , is to be associated with the control  $u$  and the second,  $J_2$ , is for the disturbance  $w$ . For notational purposes, when the arguments of the state, control, or disturbance are missing, it should be assumed that they are just suppressed. The players' cost functions are given by

$$J_1(t, x, u, w) = \int_t^{t_f} L_1(\tau, x, u, w) d\tau + \psi_1(x_f) \quad (5)$$

$$J_2(t, x, u, w) = \int_t^{t_f} L_2(\tau, x, u, w) d\tau + \psi_2(x_f) \quad (6)$$

where  $L_1, L_2$  are the running cost functions,  $\psi_1, \psi_2$  are the terminal cost functions for each player respectively, and  $x(t_f) = x_f$ . Assume the running cost  $L_i$  satisfies a polynomial growth condition  $\|L_i(t, x, u, w)\| \leq k_i(1 + \|x\|^{c_i} + \|u\|^{c_i} + \|w\|^{c_i})$  and the terminal cost  $\psi_i$  satisfies a polynomial growth condition  $\|\psi_i(t, x)\| \leq k_i(1 + \|x\|_i^{c_i})$ , where  $k_i, c_i$  are some constants and for  $i = 1, 2$ . The game to be considered here is one in which the first player, the control  $u$ , wishes to minimize a performance index

$$\phi_1(t, x, u, w) = \text{Var}_{tx}\{J_1(t, x, u, w)\} \quad (7)$$

where  $\text{Var}_{tx} := E_{tx}\{J_1^2(t, x, u, w)\} - E_{tx}^2\{J_1(t, x, u, w)\}$ . Similarly, the second player, the disturbance  $w$ , wishes to minimize

$$\phi_2(t, x, u, w) = \text{Var}_{tx}\{J_2(t, x, u, w)\} \quad (8)$$

where both of their cost's mean values are constrained.

Because both players will be assumed to have full feedback information available to them,  $\mathcal{U}_F$  will be the information pattern for the control and  $\mathcal{W}_F$  will be the information pattern for the disturbance. Thus,  $\mathcal{U}_F$  is the class of all feedback strategies  $\mu$  already described, and similarly for  $\mathcal{W}_F$ . Now we define what is meant by a Nash equilibrium solution to the game.

*Definition 1:* The pair  $(\mu^*, v^*)$  is a Nash equilibrium solution if it satisfies the inequalities

$$\phi_1(0, x, \mu^*, v^*) \leq \phi_1(0, x, \mu, v^*)$$

$$\phi_2(0, x, \mu^*, v^*) \leq \phi_2(0, x, \mu^*, v)$$

$\forall \mu \in \mathcal{U}_F$  and  $\forall v \in \mathcal{W}_F$ .

Now let  $V_1(t, x; \mu, \nu) = E_{tx}\{J_1(t, x, u, w)\}$  and  $V_2(t, x; \mu, \nu) = E_{tx}\{J_1^2(t, x, u, w)\}$  be the first and second moments of the cost function  $J_1(t, x, u, w)$ . Likewise, let  $\bar{V}_1(t, x; \mu, \nu) = E_{tx}\{J_2(t, x, u, w)\}$  and  $\bar{V}_2(t, x; \mu, \nu) = E_{tx}\{J_2^2(t, x, u, w)\}$ . Furthermore, let  $\mu^*$  denote the Nash equilibrium solution for the control and  $\nu^*$  be likewise for  $w$  in what follows.

*Definition 2:* A function  $M : \bar{Q}_0 \rightarrow \mathbb{R}^+$  is an admissible mean control cost function if there exists an admissible strategy  $\mu$  such that  $M(t, x) = V_1(t, x; \mu, \nu^*)$  for  $t \in T, x \in \mathbb{R}^n$ . Similarly,  $\bar{M} : \bar{Q}_0 \rightarrow \mathbb{R}^+$  is an admissible mean disturbance cost function if there exist  $\nu \in \mathcal{W}_F$  such that  $\bar{M}(t, x) = \bar{V}_1(t, x; \mu^*, \nu)$ .

From now on we shall assume that  $M$  and  $\bar{M}$  are admissible mean cost functions.

*Definition 3:*  $M$  defines a class of admissible strategies  $\mathcal{U}_M$  such that  $\mu \in \mathcal{U}_M$  if and only if the strategy  $\mu$  is admissible and satisfies Definition 2. Similarly for  $\bar{M}$ , there is defined a class of admissible strategies  $\mathcal{W}_M$ .

*Definition 4:* An MCV control strategy  $\mu^* \in \mathcal{U}_M$  is one that minimizes the second moment, i.e.  $V_2(t, x, \mu^*, \nu^*) = V_2(t, x) \leq V_2(t, x, \mu, \nu^*)$  for  $t \in T, x \in \mathbb{R}^n, \nu^* \in \mathcal{W}_F$ , where  $\mu \in \mathcal{U}_M$ . Furthermore the variance is found through  $V(t, x) = V_2(t, x) - M^2(t, x)$ . The disturbance's variance is likewise defined and denoted as  $\bar{V}(t, x)$ , with the Nash equilibrium solution being  $\nu^*$ .  $V_2$  is referred to as the second moment value function and  $V$  is the variance value function.

## B. Nonlinear Nash Solution

To begin this section, several lemmas that will be used in the proof of the Nash equilibrium strategies will be given. The first lemma will help by providing a necessary condition for the mean of the cost function.

*Lemma 1:* Let  $M, \bar{M} \in C_p^{1,2}(\bar{Q}_0)$  be admissible mean cost functions for the control and disturbance, respectively. Also let  $\mu$  and  $\nu$  be admissible control strategies such that they satisfy Definition 2.

Under these assumptions the admissible mean cost function  $M$  satisfies

$$\mathcal{O}^{\mu, \nu^*} M(t, x) + L_1(t, x, \mu, \nu^*) = 0 \quad (9)$$

and similarly for  $\bar{M}$

$$\mathcal{O}^{\mu^*, \nu} \bar{M}(t, x) + L_2(t, x, \mu^*, \nu) = 0 \quad (10)$$

where  $M(t_f, x_f) = \psi_1(x_f)$  and  $\bar{M}(t_f, x_f) = \psi_2(x_f)$ .

Now we have the following Verification Lemma for the mean of the cost function. It provides sufficient conditions for the mean value function. Here the set  $Q$  is to be an open subset of  $Q_0$ .

*Lemma 2 (Verification Lemma):* Let  $M \in C_p^{1,2}(Q) \cap C(\bar{Q})$  be a solution to

$$\mathcal{O}^{\mu, \nu} M(t, x) + L_1(t, x, \mu, \nu^*) = 0 \quad (11)$$

with boundary condition  $M(t_f, x_f) = \psi_1(x_f)$ . Then  $M(t, x) = V_1(t, x; \mu, \nu^*)$  for all  $\mu \in \mathcal{U}_M$ . Also if  $\bar{M} \in C_p^{1,2}(Q) \cap C(\bar{Q})$  is a solution to

$$\mathcal{O}^{\mu^*, \nu} \bar{M}(t, x) + L_2(t, x, \mu^*, \nu) = 0 \quad (12)$$

then  $\bar{M}(t, x) = \bar{V}(t, x; \mu^*, \nu)$  for all  $\nu \in \mathcal{V}_M$ .

*Proof.* Since  $M \in C_p^{1,2}(\bar{Q}_0)$  and is real valued, we can use the Dynkin formula

$$M(t, x) = E_{tx} \left\{ \int_t^{t_f} -\mathcal{O}^{\mu, \nu} M(s, x(s)) ds \right\} + E_{tx} \{ \psi_1(x_f) \}. \quad (13)$$

By (11) we have  $-\mathcal{O}^{\mu, \nu} M(t, x) = L_1(t, x, \mu, \nu^*)$ . Substituting this in (13), we see that  $M(t, x) = E_{tx} \{ J_1(t, x, \mu, \nu^*) \}$ . This is indeed the first cumulant of the cost, so  $M(t, x) = V_1(t, x; \mu, \nu^*)$ . The proof for  $\bar{M}$  simply follows the proof for  $M$ .  $\square$

Now that we have the results for the mean of the costs, we have the following lemmas for the second moment of the costs.

*Lemma 3:* Let  $M \in C_p^{1,2}(\bar{Q}_0)$  be an admissible mean cost function, with a class of admissible mean strategies  $\mathcal{U}_M$ . If  $\mu^*$  and  $V_2(t, x) \in C_p^{1,2}(\bar{Q}_0)$  are the control's Nash strategy and corresponding second moment value function, then they satisfy

$$\mathcal{O}^{\mu^*, \nu^*} V_2(t, x) + 2M(t, x)L_1(t, x, \mu^*, \nu^*) = 0 \quad (14)$$

where the disturbance's Nash strategy is played. Likewise, for the disturbance's admissible mean cost function  $\bar{M} \in C^2(Q)$ , the disturbance's Nash strategy and value function satisfy

$$\mathcal{O}^{\mu^*, \nu^*} \bar{V}_2(t, x) + 2\bar{M}(t, x)L_2(t, x, \mu^*, \nu^*) = 0 \quad (15)$$

where the control's Nash strategy is played.

*Proof.* By definition of  $V_2(t, x)$  and  $\bar{V}_2(t, x)$ , we have

$$V_2(t, x) \leq V_2(t, x; \mu, \nu^*)$$

$$\bar{V}_2(t, x) \leq \bar{V}_2(t, x; \mu^*, \nu).$$

Now, define the strategies as being their Nash strategies for times  $t + \Delta t$  to  $t_f$  and being suboptimal for times  $t$  to  $t + \Delta t$ . Then, following the work in [38] pp. 142-144, we have

$$\begin{aligned} V_2(t, x) &\leq E_{tx} \left\{ \left[ \int_t^{t+\Delta t} L_{1s} ds \right]^2 \right\} + E_{tx} \left\{ \left[ \int_t^{t+\Delta t} L_{1s} ds \right] \left[ \int_{t+\Delta t}^{t_f} L_{1s}^* ds + \psi_1(x_f) \right] \right\} \\ &\quad + E_{tx} \left\{ \left[ \int_{t+\Delta t}^{t_f} L_{1s}^* ds + \psi_1(x_f) \right]^2 \right\} \\ \bar{V}_2(t, x) &\leq E_{tx} \left\{ \left[ \int_t^{t+\Delta t} L_{2s} ds \right]^2 \right\} + E_{tx} \left\{ \left[ \int_t^{t+\Delta t} L_{2s} ds \right] \left[ \int_{t+\Delta t}^{t_f} L_{2s}^* ds + \psi_2(x_f) \right] \right\} \\ &\quad + E_{tx} \left\{ \left[ \int_{t+\Delta t}^{t_f} L_{2s}^* ds + \psi_2(x_f) \right]^2 \right\} \end{aligned}$$

where  $L_1(s, x; \mu, \mathbf{v}^*) = L_{1s}$ ,  $L_1(s, x; \mu^*, \mathbf{v}^*) = L_{1s}^*$  and similarly for  $L_2$ . From [38], page 143, using the mean value theorem and Chapman-Kolmogorov equation yields

$$\begin{aligned} V_2(t, x) &\leq \Delta t^2 E_{tx} \{ L_1^2(t^+, x^+, \mu^+, \mathbf{v}^*) \} + 2\Delta t E_{tx} \left\{ L_1(t^+, x^+, \mu^+, \mathbf{v}^*) \right. \\ &\quad \cdot \left. \left[ \int_{t+\Delta t}^{t_f} L_{1s}^* ds + \psi_1(x_f) \right] \right\} + E_{tx} \{ V_2(t + \Delta t, x(t + \Delta t)) \} \\ \bar{V}_2(t, x) &\leq \Delta t^2 E_{tx} \{ L_2^2(t^+, x^+, \mu^*, \mathbf{v}^+) \} + 2\Delta t E_{tx} \left\{ L_2(t^+, x^+, \mu^*, \mathbf{v}^+) \right. \\ &\quad \cdot \left. \left[ \int_{t+\Delta t}^{t_f} L_{2s}^* ds + \psi_1(x_f) \right] \right\} + E_{tx} \{ \bar{V}_2(t + \Delta t, x(t + \Delta t)) \}. \end{aligned}$$

Notice the term  $E_{tx} \{ \bar{V}_2(t + \Delta t, x(t + \Delta t)) \}$ . This can be seen as a terminal condition in the Dynkin formula. With the application of the Dynkin formula, we can now write

$$\begin{aligned} V_2(t, x) &\leq \Delta t^2 E_{tx} \{ L_1^2(t^+, x^+, \mu^+, \mathbf{v}^*) \} + 2\Delta t E_{tx} \left\{ L_1(t^+, x^+, \mu^+, \mathbf{v}^*) \right. \\ &\quad \cdot \left. \left[ \int_{t+\Delta t}^{t_f} L_{1s}^* ds + \psi_1(x_f) \right] \right\} + E_{tx} \left\{ \int_t^{t+\Delta t} \mathcal{O}^{\mu, \mathbf{v}^*} V_2(s, x(s)) ds \right\} \\ &\quad + V_2(t, x) \\ \bar{V}_2(t, x) &\leq \Delta t^2 E_{tx} \{ L_2^2(t^+, x^+, \mu^*, \mathbf{v}^+) \} + 2\Delta t E_{tx} \left\{ L_2(t^+, x^+, \mu^*, \mathbf{v}^+) \right. \\ &\quad \cdot \left. \left[ \int_{t+\Delta t}^{t_f} L_{2s}^* ds + \psi_1(x_f) \right] \right\} + E_{tx} \left\{ \int_t^{t+\Delta t} \mathcal{O}^{\mu^*, \mathbf{v}^+} \bar{V}_2(s, x(s)) ds \right\} \\ &\quad + \bar{V}_2(t, x). \end{aligned}$$

As it was shown in [38], page 143, if we let  $\Delta t$  go to zero, then we are left with

$$0 \leq \mathcal{O}^{\mu, v^*} V_2(t, x) + 2M(t, x)L_1(t, x, \mu(t, x), v^*(t, x))$$

$$0 \leq \mathcal{O}^{\mu, v^*} \bar{V}_2(t, x) + 2\bar{M}(t, x)L_2(t, x, \mu^*(t, x), v(t, x))$$

which is almost the desired result. But notice, if the Nash equilibrium strategies  $\mu^*$  and  $v^*$  are played, then the inequalities become equalities and the theorem is proved.  $\square$

*Lemma 4 (Verification Lemma):* Let  $V_2 \in C_p^{1,2}(\mathcal{Q}) \cap C(\bar{\mathcal{Q}})$  be a nonnegative solution to the partial differential equation

$$\min_{\mu \in \mathcal{U}_M} \left\{ \mathcal{O}^{\mu, v^*} V_2(t, x) + 2M(t, x)L_1(t, x, \mu, v^*) \right\} = 0 \quad (16)$$

with boundary condition  $V_2(t_f, x_f) = \psi_1^2(x_f)$ . Then  $V_2(t, x) \leq V_2(t, x; \mu, v^*)$  for every  $\mu \in \mathcal{U}_M$ , and  $(t, x) \in \bar{\mathcal{Q}}_0$ . If  $\mu$  also satisfies

$$\begin{aligned} \min_{\tilde{\mu} \in \mathcal{U}_M} \left\{ \mathcal{O}^{\tilde{\mu}, v^*} V_2(t, x) + 2M(t, x)L_1(t, x, \tilde{\mu}, v^*) \right\} \\ = \mathcal{O}^{\mu, v^*} V_2(t, x) + 2M(t, x)L_1(t, x, \mu, v^*) \end{aligned} \quad (17)$$

for all  $(t, x) \in \bar{\mathcal{Q}}_0$ , then  $V_2(t, x) = V_2(t, x; \mu, v^*)$ . Similarly for the disturbance, if  $v \in \mathcal{V}_M$  is the minimizing argument of

$$\min_{v \in \mathcal{V}_M} \left\{ \mathcal{O}^{\mu^*, v} \bar{V}_2(t, x) + 2\bar{M}(t, x)L_2(t, x, \mu^*, v) \right\} = 0 \quad (18)$$

then  $\bar{V}_2(t, x) = \bar{V}_2(t, x; \mu^*, v)$ .

*Proof.* This proof will closely follow the proof of theorem 4.2 in [37]. Assume that  $\mu$  satisfies (17) and apply the Dynkin formula:

$$V_2(t, x) = E_{tx} \left\{ \int_t^{t_f} -\mathcal{O}^{\mu, v^*} V_2(s, x) ds + \psi_1^2(x_f) \right\}. \quad (19)$$

From (16) we have the following

$$V_2(t, x) \leq E_{tx} \left\{ \int_t^{t_f} 2L_1(s, x, \mu(s, x), v^*(s, x)) M(s, x(s)) ds \right\} + E_{tx} \{ \psi_1^2(x_f) \}.$$

Now let  $L_\tau = L_1(\tau, x(\tau), \mu(\tau, x(\tau)), v^*(\tau, x(\tau)))$ ; from Lemma 1 we have

$$\begin{aligned} E_{tx} \left\{ \int_t^{t_f} 2L_1(s, x, \mu(s, x), v^*(s, x)) M(s, x(s)) ds \right\} + E_{tx} \{ \psi_1^2(x_f) \} \\ = E_{tx} \left\{ \int_t^{t_f} 2L_s E_{sx} \left\{ \int_s^{t_f} L_r dr + \psi_1(x_f) \right\} ds \right\} + E_{tx} \{ \psi_1^2(x_f) \} \\ = E_{tx} \left\{ \int_t^{t_f} E_{sx} \left\{ 2L_s \int_s^{t_f} L_r dr + 2L_s \psi_1(x_f) \right\} ds \right\} + E_{tx} \{ \psi_1^2(x_f) \} \end{aligned}$$

and, by interchanging the expectation and the integral,

$$\begin{aligned}
& E_{tx} \left\{ \int_t^{t_f} 2L_1(s, x, \mu(s, x), \mathbf{v}^*(s, x)) M(s, x(s)) ds \right\} + E_{tx} \{ \psi_1^2(x_f) \} \\
&= \int_t^{t_f} E_{tx} \left\{ E_{sx} \left\{ 2L_s \int_s^{t_f} L_r dr + 2L_s \psi_1(x_f) \right\} \right\} ds + E_{tx} \{ \psi_1^2(x_f) \} \\
&= \int_t^{t_f} E_{tx} \left\{ \left[ 2L_s \int_s^{t_f} L_r dr + 2L_s \psi_1(x_f) \right] \right\} ds + E_{tx} \{ \psi_1^2(x_f) \} \\
&= E_{tx} \left\{ \int_t^{t_f} \left[ 2L_s \int_s^{t_f} L_r dr + 2L_s \psi_1(x_f) \right] ds \right\} + E_{tx} \{ \psi_1^2(x_f) \} \\
&= E_{tx} \left\{ \int_t^{t_f} L_s ds \int_t^{t_f} L_r dr + 2 \int_t^{t_f} L_s \psi_1(x_f) ds \right\} + E_{tx} \{ \psi_1^2(x_f) \} \\
&= E_{tx} \left\{ \left[ \int_t^{t_f} L_s ds + \psi_1(x_f) \right]^2 \right\} = V_2(t, x; \mu, \mathbf{v}^*).
\end{aligned}$$

From the above analysis we can see that if  $\mu^*$  is the minimizing solution then the inequality becomes an equality.

The proof for the case of the disturbance may be completed by following the proof for the control.  $\square$

From these lemmas, it is possible to begin discussion of the Nash equilibrium solution. The following two theorems provide sufficient and necessary conditions for the Nash equilibrium solution.

*Theorem 1:* Consider the two player game described by (1), (7), and (8). Let  $M$  be an admissible mean cost function,  $M \in C_p^{1,2}(\mathcal{Q}) \cap C(\bar{\mathcal{Q}})$ , with an associated  $\mathcal{U}_M$ . Also consider the function  $V \in C_p^{1,2}(\mathcal{Q}) \cap C(\bar{\mathcal{Q}})$  that is a solution to

$$\min_{\mu \in \mathcal{U}_M} \left\{ \mathcal{O}^{\mu, \mathbf{v}^*} V(t, x) + \left| \frac{\partial M}{\partial x}(t, x) \right|_{\sigma(t, x)W(t)\sigma'(t, x)}^2 \right\} = 0 \quad (20)$$

with  $V(t_f, x_f) = 0$  and the function  $\bar{V} \in C_p^{1,2}(\mathcal{Q}) \cap C(\bar{\mathcal{Q}})$  that satisfies

$$\min_{\mathbf{v} \in \mathcal{U}_M} \left\{ \mathcal{O}^{\mu^*, \mathbf{v}} \bar{V}(t, x) + \left| \frac{\partial \bar{M}}{\partial x}(t, x) \right|_{\sigma(t, x)W(t)\sigma'(t, x)}^2 \right\} = 0 \quad (21)$$

with  $\bar{V}(t_f, x_f) = 0$ . If the strategies  $\mu^*$  and  $\mathbf{v}^*$  are the minimizing arguments of (20) and (21), then the pair  $(\mu^*, \mathbf{v}^*)$  constitutes a Nash equilibrium solution.

*Proof.* Let the disturbance play its Nash equilibrium strategy  $\mathbf{v}^*$ . Now assume  $M$  is an admissible mean cost function such that  $M^2 \in C_p^{1,2}(\mathcal{Q}) \cap C(\bar{\mathcal{Q}})$  and assume  $V \in C_p^{1,2}(\mathcal{Q}) \cap C(\bar{\mathcal{Q}})$  satisfies (20). Let  $\mu \in \mathcal{U}_M$  be an admissible control strategy which may or may not be the minimal strategy in (20). Recall that if the control strategy  $\mu$  is in the class of admissible mean strategies  $\mathcal{U}_M$ , then it is such that

$$\mathcal{O}^{\mu, \mathbf{v}^*} M(t, x) + L_1(t, x, \mu, \mathbf{v}^*) = 0 \quad (22)$$

where  $M(t_f, x_f) = \psi_1(x_f)$ . Because  $\mu$  may or may not be optimal we have

$$\mathcal{O}^{\mu, \nu^*} V(t, x) + \left| \frac{\partial M}{\partial x}(t, x) \right|_{\sigma(t, x)W(t)\sigma'(t, x)}^2 \geq 0 \quad (23)$$

where  $V(t_f, x_f) = 0$ . Manipulating the above equation yields

$$V(t, x) = E_{tx} \left\{ \int_t^{t_f} -\mathcal{O}^{\mu, \nu^*} V(s, x) ds \right\} \leq E_{tx} \left\{ \int_t^{t_f} \left| \frac{\partial M}{\partial x}(s, x) \right|_{\sigma(s, x)W(s)\sigma'(s, x)}^2 ds \right\} \quad (24)$$

where once again the Dynkin formula is used. Recall that in order for  $V(t, x)$  to be a value function for the variance, it must be such that  $V(t, x) = V_2(t, x) - M^2(t, x)$ . But since  $M^2 \in C_p^{1,2}(\mathcal{Q}) \cap C(\bar{\mathcal{Q}})$ , then  $V_2 \in C_p^{1,2}(\mathcal{Q}) \cap C(\bar{\mathcal{Q}})$ . Using the definition of variance we have

$$V_2(t, x) - M^2(t, x) \leq E_{tx} \left\{ \int_t^{t_f} \left| \frac{\partial M}{\partial x}(s, x) \right|_{\sigma W \sigma'}^2 ds \right\} \quad (25)$$

which yields

$$\begin{aligned} V_2(t, x) &\leq E_{tx} \left\{ \int_t^{t_f} \left| \frac{\partial M}{\partial x}(s, x) \right|_{\sigma W \sigma'}^2 ds \right\} \\ &\quad - E_{tx} \left\{ \int_t^{t_f} \mathcal{O}^{\mu, \nu^*} M^2(s, x) ds \right\} + E_{tx} \{ \psi_1^2(x_f) \} \end{aligned} \quad (26)$$

by the application of the Dynkin formula to the function  $M^2(t, x)$ . From Lemma 4 we also have

$$V_2(t, x) \leq E_{tx} \left\{ \int_t^{t_f} 2M(s, x)L_1(s, x, \mu, \nu^*) ds \right\} + E_{tx} \{ \psi_1^2(x_f) \} \quad (27)$$

with another usage of the Dynkin formula. If we can show that the right members of (26) and (27) are equal, then we can employ the techniques of Lemma 4 to obtain the desired result. To do so, we examine the equality

$$\mathcal{O}^{\mu, \nu^*} M^2(t, x) + 2M(t, x)L_1(t, x, \mu, \nu^*) = \left| \frac{\partial M}{\partial x}(t, x) \right|_{\sigma W \sigma'}^2 \quad (28)$$

for  $(t, x) \in \mathcal{Q}$ . The first step in this pursuit is to let  $\mathcal{O}^{\mu, \nu} = \mathcal{O}_1^{\mu, \nu} + \mathcal{O}_2$  where

$$\begin{aligned} \mathcal{O}_1^{\mu, \nu} &= \frac{\partial}{\partial t} + f'(t, x, \mu(t, x), \nu(t, x)) \frac{\partial}{\partial x} \\ \mathcal{O}_2 &= \frac{1}{2} \text{tr} \left( \sigma(t, x)W(t)\sigma'(t, x) \frac{\partial^2}{\partial x^2} \right) \end{aligned}$$

and with this definition we have for  $\mathcal{O}^{\mu, \nu^*} M^2(t, x)$

$$\mathcal{O}^{\mu, \nu^*} M^2(t, x) = 2M(t, x)\mathcal{O}_1^{\mu, \nu^*} M(t, x) + \mathcal{O}_2 M^2(t, x).$$

Recall that  $M$  is an admissible mean cost function, therefore  $-\mathcal{O}^{\mu, \nu^*} M(t, x) = L_1(t, x, \mu, \nu^*)$ . With these two observations, (28) reduces to

$$\mathcal{O}_2 M^2(t, x) - 2M(t, x)\mathcal{O}_2 M(t, x) = \left| \frac{\partial M}{\partial x}(t, x) \right|_{\sigma W \sigma'}^2. \quad (29)$$

We wish to show that (29) holds. Recall that

$$\frac{\partial^2 M^2}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial M^2}{\partial x} \right)'$$

so that the left members of (29) becomes

$$\frac{1}{2} \text{tr} \left( \sigma W \sigma' \left[ \frac{\partial^2 M^2}{\partial x^2} - 2M(t,x) \frac{\partial^2 M}{\partial x^2} \right] \right) = \text{tr} \left( \sigma W \sigma' \left( \frac{\partial M}{\partial x} \right) \left( \frac{\partial M}{\partial x} \right)' \right) \quad (30)$$

where the arguments are suppressed. But notice that the right members in (30) equal the right members of (29).

Therefore the equality (28) is established. This results in (26) and (27) being equivalent, which in turn says that

$V_2(t,x) \leq V_2(t,x;\mu, \mathbf{v}^*)$  for all  $\mu \in \mathcal{U}_M$  and  $(t,x) \in \mathcal{Q}$ . With another application of Definition 4 we see that  $V(t,x) \leq$

$V(t,x;\mu, \mathbf{v}^*)$  for all  $\mu \in \mathcal{U}_M$  and  $(t,x) \in \mathcal{Q}$  and therefore we have reached the desired result. Note that if indeed

$\mu = \mu^*$ , the inequalities in the second part of the proof become equalities and therefore  $V(t,x) = V(t,x;\mu^*, \mathbf{v}^*)$ .

For (21), the proof is the same, except the minimization takes place for all  $\mathbf{v} \in \mathcal{W}_M$  instead of  $\mu \in \mathcal{U}_M$ , as well as

$\bar{V}$  and  $\bar{M}$  take the place of  $V$  and  $M$ .  $\square$

*Theorem 2:* Let  $M$  be an admissible mean cost function,  $M \in C_p^{1,2}(\bar{\mathcal{Q}}_0)$ , with an associated  $\mathcal{U}_M$  and similarly for  $\bar{M}$  and  $\mathcal{W}_M$ . If  $(\mu^*, \mathbf{v}^*)$  is the Nash equilibrium pair and  $V(t,x) \in C_p^{1,2}(\bar{\mathcal{Q}}_0)$ ,  $\bar{V}(t,x) \in C_p^{1,2}(\bar{\mathcal{Q}}_0)$  are the corresponding variance value functions, then they satisfy the equations

$$\mathcal{O}^{\mu^*, \mathbf{v}^*} V(t,x) + \left| \frac{\partial M}{\partial x}(t,x) \right|_{\sigma(t,x)W(t)\sigma'(t,x)}^2 = 0 \quad (31)$$

$$\mathcal{O}^{\mu^*, \mathbf{v}^*} \bar{V}(t,x) + \left| \frac{\partial \bar{M}}{\partial x}(t,x) \right|_{\sigma(t,x)W(t)\sigma'(t,x)}^2 = 0 \quad (32)$$

with  $\bar{V}(t_f, x_f) = 0$  and  $V(t_f, x_f) = 0$ .

*Proof.* By definition,  $V(t,x) = V_2(t,x) - M^2(t,x)$  and likewise for the disturbance we have  $\bar{V}(t,x) = \bar{V}_2(t,x) - \bar{M}^2(t,x)$ . From Lemma 3, we have

$$\mathcal{O}^{\mu^*, \mathbf{v}^*} [V(t,x) + M^2(t,x)] + 2M(t,x)L_1(t,x;\mu^*, \mathbf{v}^*) = 0$$

$$\mathcal{O}^{\mu^*, \mathbf{v}^*} [\bar{V}(t,x) + \bar{M}^2(t,x)] + 2\bar{M}(t,x)L_2(t,x;\mu^*, \mathbf{v}^*) = 0.$$

However, it was shown in the proof for Theorem 1, that

$$\begin{aligned} \mathcal{O}^{\mu^*, \mathbf{v}^*} [M^2(t,x)] + 2M(t,x)L_1(t,x;\mu^*, \mathbf{v}^*) &= \left| \frac{\partial M}{\partial x}(t,x) \right|_{\sigma(t,x)W(t)\sigma'(t,x)}^2 \\ \mathcal{O}^{\mu^*, \mathbf{v}^*} [\bar{M}^2(t,x)] + 2\bar{M}(t,x)L_2(t,x;\mu^*, \mathbf{v}^*) &= \left| \frac{\partial \bar{M}}{\partial x}(t,x) \right|_{\sigma(t,x)W(t)\sigma'(t,x)}^2. \end{aligned}$$

With the linearity of the operator  $\mathcal{O}^{\mu, \nu}$  and through substitution, the theorem is proved.  $\square$

*Remark 1:* Even though the treatment of the Nash game has been for two players, this problem can be extended to  $N$  players. Due to space constraints, this isn't included. However, for this discussion please see [13].

#### IV. ZERO SUM (MINIMAX) GAME

##### A. Problem Definition

So far, the discussion has been limited to the case in which both players have different cost functions, and each player wishes to minimize their performance index when the other player has played their best strategy. Now, the discussion will switch gears to the zero-sum game, in which the players both are concerned with the same performance index, only the control wishes to minimize it, while the disturbance wishes to maximize it.

For the performance index, the control  $u$  wishes to minimize the variance of the cost, while the disturbance  $w$  wishes to maximize this quantity, with the mean value being constrained. As in the non-zero sum case, when the arguments of the state, control, or disturbance  $w$  are missing, it should be assumed that they are just suppressed. The players' cost function is given by

$$J(t, x, u, w) = \int_t^{t_f} L(\tau, x, u, w) d\tau + \psi(x_f) \quad (33)$$

where  $L$  is the running cost functions and  $\psi$  is the terminal cost function, respectively, and  $x(t_f) = x_f$ . The running cost and terminal cost both are assumed to satisfy a polynomial growth condition. Because the variance is the cumulant of interest, the performance is given by

$$\phi(t, x, u, w) = \text{Var}_{ix}\{J(t, x, u, w)\} \quad (34)$$

where the mean is to be constrained. The players in the zero-sum game will also have full feedback information available, and  $\mathcal{U}_F$  and  $\mathcal{W}_F$  are defined similar to before. This allows us to define what is meant by a zero sum equilibrium solution.

*Definition 5:* The pair  $(\mu^*, \nu^*)$  is a zero sum equilibrium solution if it satisfies the inequality

$$\phi(0, x, \mu^*, \nu) \leq \phi(0, x, \mu^*, \nu^*) \leq \phi(0, x, \mu, \nu^*)$$

$\forall \mu \in \mathcal{U}_F$  and  $\forall \nu \in \mathcal{W}_F$ .

Let  $V_1(t, x; \mu, \nu) = E_{tx}\{J(t, x, u, w)\}$  and  $V_2(t, x; \mu, \nu) = E_{tx}\{J^2(t, x, u, w)\}$  be the first and second moments of the cost function  $J(t, x, u, w)$ .

*Definition 6:* A function  $M: \bar{Q}_0 \rightarrow \mathbb{R}^+$  is an admissible mean cost function if there exist admissible strategies  $\mu$  and  $\nu$  such that  $M(t, x) = V_1(t, x; \mu, \nu)$  for  $t \in T, x \in \mathbb{R}^n$ .

From now on we shall assume that  $M$  is an admissible mean cost function.

*Definition 7:*  $M$  defines a class of admissible strategies  $\mathcal{U}_M$  and  $\mathcal{W}_M$  such that  $\mu \in \mathcal{U}_M$  and  $\nu \in \mathcal{W}_M$  if and only if the strategies are admissible and satisfy Definition 6.

*Definition 8:* The optimal control strategy  $\mu^* \in \mathcal{U}_M$  is one that minimizes the second moment, that is  $V_2(t, x, \mu^*, \nu^*) = V_2(t, x) \leq V_2(t, x, \mu, \nu^*)$  for  $t \in T, x \in \mathbb{R}^n, \nu^* \in \mathcal{W}_F$ , where  $\mu \in \mathcal{U}_M$ . Furthermore the variance is found through  $V(t, x) = V_2(t, x) - M^2(t, x)$ . Likewise, the optimal disturbance strategy  $\nu^* \in \mathcal{W}_M$  maximizes  $V_2(t, x, \mu^*, \nu^*) = V_2(t, x) \leq V_2(t, x, \mu^*, \nu)$ .

### B. Zero Sum Equilibrium Solution

Next, we will give the zero sum versions of the lemmas and theorems given in the non-zero sum game.

*Lemma 5:* Let  $M \in C_p^{1,2}(\bar{Q}_0)$  be an admissible mean cost function and  $\mu$  and  $\nu$  be admissible control strategies such that they satisfy Definition 6. Under these assumptions the admissible mean cost function  $M$  satisfies

$$\mathcal{O}^{\mu, \nu} M(t, x) + L(t, x, \mu, \nu) = 0 \quad (35)$$

where  $M(t_f, x_f) = \psi(x_f)$ .

With the necessary condition result, a sufficient condition may be desirable.

*Lemma 6 (Verification Lemma):* Let  $M \in C_p^{1,2}(Q) \cap C(\bar{Q})$  be a solution to

$$\mathcal{O}^{\mu, \nu} M(t, x) + L(t, x, \mu, \nu) = 0 \quad (36)$$

with boundary condition  $M(t_f, x_f) = \psi(x_f)$ . Then  $M(t, x) = V_1(t, x; \mu, \nu)$  for all  $\mu \in \mathcal{U}_M$  and  $\nu \in \mathcal{W}_M$ .

*Proof:* Let  $M \in C_p^{1,2}(\bar{Q}_0)$  and be real valued. The Dynkin formula can be used to give

$$M(t, x) = E_{tx} \left\{ \int_t^{t_f} -\mathcal{O}^{\mu, \nu} M(s, x(s)) ds \right\} + E_{tx} \{ \psi(x_f) \}. \quad (37)$$

But by (36) we have  $-\mathcal{O}^{\mu, \nu} M(t, x) = L(t, x, \mu, \nu)$ . With some substitution, we find  $M(t, x) = V_1(t, x; \mu, \nu)$  and the lemma is proved.  $\square$

*Lemma 7:* Let  $M \in C_p^{1,2}(\bar{Q}_0)$  be an admissible mean cost function, with a class of admissible mean strategies  $\mathcal{U}_M$  and  $\mathcal{V}_M$ . If  $\mu^*$ ,  $\nu^*$ , and  $V_2(t, x) \in C_p^{1,2}(\bar{Q}_0)$  are the zero sum equilibrium strategies and corresponding second moment value function, then they satisfy

$$\mathcal{O}^{\mu^*, \nu^*} V_2(t, x) + 2M(t, x)L(t, x, \mu^*, \nu^*) = 0. \quad (38)$$

*Proof.* For the zero sum game, the proof will follow the non-zero sum game, Lemma 3. Recall, that the proof followed that of [38], pp. 142-144, closely. Both strategies will be defined as they were in non-zero sum game, where they play their optimal strategy for time  $t + \Delta t$  to  $t_f$  and suboptimal for time  $t$  to  $t + \Delta t$ . By definition of  $V_2(t, x)$  we have

$$V_2(t, x) \leq V_2(t, x; \mu, \nu^*)$$

$$V_2(t, x) \geq V_2(t, x; \mu^*, \nu).$$

Notice that for the first inequality, it is very easy to see that this proof is almost the same as the non-zero sum game except that instead of  $L_1$  we now have  $L$ . For the second inequality, the proof is also the same as that in the non-zero sum game, except the less than or equal operator is replaced by greater than or equal.  $\square$

Next, we have the following Verification Lemma for the second moment of the cost.

*Lemma 8 (Verification Lemma):* Let  $V_2 \in C_p^{1,2}(Q) \cap C(\bar{Q})$  be a nonnegative solution to the partial differential equation

$$\begin{aligned} \min_{\mu \in \mathcal{U}_M} \max_{\nu \in \mathcal{V}_M} \{ \mathcal{O}^{\mu, \nu} V_2(t, x) + 2M(t, x)L(t, x, \mu, \nu) \} = \\ \max_{\nu \in \mathcal{V}_M} \min_{\mu \in \mathcal{U}_M} \{ \mathcal{O}^{\mu, \nu} V_2(t, x) + 2M(t, x)L(t, x, \mu, \nu) \} = \\ \mathcal{O}^{\mu^*, \nu^*} V_2(t, x) + 2M(t, x)L(t, x, \mu^*, \nu^*) = 0 \end{aligned} \quad (39)$$

with boundary condition  $V_2(t_f, x_f) = \psi^2(x_f)$ . Then  $V_2(t, x; \mu^*, \nu) \leq V_2(t, x) \leq V_2(t, x; \mu, \nu^*)$  for every  $\mu \in \mathcal{U}_M$ ,  $\nu \in \mathcal{V}_M$ , and  $(t, x) \in \bar{Q}_0$ .

*Proof.* To do this proof, we will follow the proof of Theorem 4.2 in [37], as we did for the corresponding proof in the non-zero sum game. Note that if the disturbance's maximizing solution is played, the proof follows the proof for Lemma 4. Therefore, assume that the control has played its minimizing solution,  $\mu^*$ , and apply the Dynkin formula:

$$V_2(t, x) = E_{tx} \left\{ \int_t^{t_f} -\mathcal{O}^{\mu^*, \nu} V_2(s, x) ds + \psi^2(x_f) \right\}. \quad (40)$$

From (39) we have the following

$$V_2(t, x) \geq E_{tx} \left\{ \int_t^{t_f} 2L(s, x, \mu^*(s, x), v(s, x))M(s, x(s))ds \right\} + E_{tx} \{ \psi^2(x_f) \}.$$

Let  $L_\tau = L(\tau, x(\tau), \mu^*(\tau, x(\tau)), v(\tau, x(\tau)))$ ; from Lemma 5 we have

$$\begin{aligned} & E_{tx} \left\{ \int_t^{t_f} 2L(s, x, \mu^*(s, x), v(s, x))M(s, x(s))ds \right\} + E_{tx} \{ \psi^2(x_f) \} \\ &= E_{tx} \left\{ \int_t^{t_f} 2L_s E_{sx} \left\{ \int_s^{t_f} L_r dr + \psi(x_f) \right\} ds \right\} + E_{tx} \{ \psi^2(x_f) \} \\ &= E_{tx} \left\{ \int_t^{t_f} L_s ds \int_t^{t_f} L_r dr + 2 \int_t^{t_f} L_s \psi(x_f) ds \right\} + E_{tx} \{ \psi^2(x_f) \} \end{aligned}$$

where the steps for the reduction were the same as in Lemma 4. Therefore

$$E_{tx} \left\{ \int_t^{t_f} 2L_s M(s, x(s)) ds \right\} = E_{tx} \left\{ \left[ \int_t^{t_f} L_s ds + \psi(x_f) \right]^2 \right\} \leq V_2(t, x; \mu, v^*).$$

Note that if the disturbance's equilibrium strategy, that is, its maximum strategy is played, then the inequalities become equalities.  $\square$

With this work, we can now give sufficient conditions for the second cumulant problem.

*Theorem 3:* Consider the two player game described by (1) and (33). Let  $M$  be an admissible mean cost function,  $M \in C_p^{1,2}(Q) \cap C(\bar{Q})$ , with associated  $\mathcal{U}_M$  and  $\mathcal{W}_M$ . Also consider the function  $V \in C_p^{1,2}(Q) \cap C(\bar{Q})$  that is a solution to

$$\begin{aligned} & \max_{v \in \mathcal{W}_M} \min_{\mu \in \mathcal{U}_M} \left\{ \mathcal{O}^{\mu, v} V(t, x) + \left| \frac{\partial M}{\partial x}(t, x) \right|_{\sigma(t, x)W(t)\sigma'(t, x)}^2 \right\} = \\ & \min_{\mu \in \mathcal{U}_M} \max_{v \in \mathcal{W}_M} \left\{ \mathcal{O}^{\mu, v} V(t, x) + \left| \frac{\partial M}{\partial x}(t, x) \right|_{\sigma(t, x)W(t)\sigma'(t, x)}^2 \right\} = \\ & \mathcal{O}^{\mu^*, v^*} V(t, x) + \left| \frac{\partial M}{\partial x}(t, x) \right|_{\sigma(t, x)W(t)\sigma'(t, x)}^2 = 0 \end{aligned} \quad (41)$$

with  $V(t_f, x_f) = 0$ . If there is a solution  $V$  to this equation, with  $\mu^*$  and  $v^*$  being the minimizing and maximizing arguments, respectively, then  $V(t, x, \mu^*, v) \leq V(t, x) \leq V(t, x, \mu, v^*)$  for all  $\mu \in \mathcal{U}_M$  and  $v \in \mathcal{W}_M$ .

*Proof.* Let the control play its minimizing strategy  $\mu^*$ . Because it is assumed that  $v \in \mathcal{W}_M$ , then

$$\mathcal{O}^{\mu, v^*} M(t, x) + L(t, x, \mu, v^*) = 0 \quad (42)$$

where  $M(t_f, x_f) = \psi(x_f)$ . Because  $\mu$  may or may not be optimal we have

$$\mathcal{O}^{\mu^*, v} V(t, x) + \left| \frac{\partial M}{\partial x}(t, x) \right|_{\sigma(t, x)W(t)\sigma'(t, x)}^2 \leq 0 \quad (43)$$

where  $V(t_f, x_f) = 0$ . Manipulating the above equation yields

$$V(t, x) = E_{tx} \left\{ \int_t^{t_f} -\mathcal{O}^{\mu^*, v} V(s, x) ds \right\} \geq E_{tx} \left\{ \int_t^{t_f} \left| \frac{\partial M}{\partial x}(s, x) \right|_{\sigma(s, x)W(s)\sigma'(s, x)}^2 ds \right\} \quad (44)$$

where once again the Dynkin formula is used. Using the definition of variance we have

$$V_2(t, x) - M^2(t, x) \geq E_{tx} \left\{ \int_t^{t_f} \left| \frac{\partial M}{\partial x}(s, x) \right|_{\sigma W \sigma'}^2 ds \right\} \quad (45)$$

which yields

$$\begin{aligned} V_2(t, x) &\geq E_{tx} \left\{ \int_t^{t_f} \left| \frac{\partial M}{\partial x}(s, x) \right|_{\sigma W \sigma'}^2 ds \right\} \\ &\quad - E_{tx} \left\{ \int_t^{t_f} \mathcal{O}^{\mu, v^*} M^2(s, x) ds \right\} + E_{tx} \{ \psi^2(x_f) \} \end{aligned} \quad (46)$$

by the application of the Dynkin formula to the function  $M^2(t, x)$ . From Lemma 8 we also have

$$V_2(t, x) \geq E_{tx} \left\{ \int_t^{t_f} 2M(s, x)L(s, x, \mu^*, v) ds \right\} + E_{tx} \{ \psi^2(x_f) \} \quad (47)$$

with another usage of the Dynkin formula. Notice that by following the steps of Theorem 1, we find that

$$\mathcal{O}^{\mu, v^*} M^2(t, x) + 2M(t, x)L(t, x, \mu^*, v^*) = \left| \frac{\partial M}{\partial x}(t, x) \right|_{\sigma W \sigma'}^2 \quad (48)$$

for  $(t, x) \in \mathcal{Q}$ . This then yields (47). For the control's case, in which the disturbance has already played its maximizing solution, the proof follows very closely that of Theorem 1.  $\square$

*Theorem 4:* Let  $M \in C_p^{1,2}(\bar{\mathcal{Q}}_0)$  be an admissible mean cost function, with an associated  $\mathcal{U}_M$  and  $\mathcal{W}_M$ . If  $(\mu^*, v^*)$  is the minimax equilibrium pair and  $V(t, x) \in C_p^{1,2}(\bar{\mathcal{Q}}_0)$  is the corresponding variance value function, then they satisfy the equations

$$\mathcal{O}^{\mu^*, v^*} V(t, x) + \left| \frac{\partial M}{\partial x}(t, x) \right|_{\sigma(t, x)W(t)\sigma'(t, x)}^2 = 0 \quad (49)$$

with  $V(t_f, x_f) = 0$ .

*Proof.* The proof is very similar to that of Theorem 2 and it follows from those steps.  $\square$

## V. APPLICATION TO THE LINEAR QUADRATIC SPECIAL CASE

### A. Non-Zero Sum Case

Now we consider the case when the system given is linear. The system will be described by

$$dx(t) = (A(t)x(t) + B(t)u(t) + D(t)w(t))dt + E(t)d\xi(t) \quad (50)$$

where  $x(t_0) = x_0$ ,  $Q$  are positive semidefinite and  $R, \bar{S}$  are positive definite. Furthermore the costs will be assumed to be quadratic;

$$J_1 = \int_{t_0}^{t_f} (x'(t)Q(t)x(t) + u'(t)R(t)u(t) + w'(t)S(t)w(t))dt + x'(t_f)Q_f x(t_f)$$

$$J_2 = \int_{t_0}^{t_f} (x'(t)\bar{Q}(t)x(t) + u'(t)\bar{R}(t)u(t) + w'(t)\bar{S}(t)w(t))dt + x'(t_f)\bar{Q}_f x(t_f)$$

where  $Q_f = \bar{Q}_f = 0$ .

Let us assume that the costs are quadratic. That is  $M(t, x) = x' \mathcal{M}(t)x + m(t)$ , and similarly with  $V(t, x), \bar{M}(t, x), \bar{V}(t, x)$  where  $\mathcal{M}, \mathcal{V}, \bar{\mathcal{M}}, \bar{\mathcal{V}}$  are matrix functions of time and  $m, v, \bar{m}, \bar{v}$  are scalar functions of time.

Recall the HJB equations for the non-zero sum case. For the control, we have

$$\min_{\mu \in \mathcal{U}_M} \left\{ \gamma_1 [x' \mathcal{M} x + \dot{m} + 2(Ax + B\mu + Dv^*)' \mathcal{M} x + x' Q x \right. \\ \left. + \mu' R \mu + v' S v] + tr(EWE'(\gamma_1 \mathcal{M} + \gamma_2 \mathcal{V})) \right. \\ \left. + \gamma_2 [x' \mathcal{V} x + \dot{v} + 2(Ax + B\mu + Dv^*)' \mathcal{V} x + 4 \mathcal{M} EWE' \mathcal{M}] \right\} = 0;$$

and minimizing this gives

$$u^*(t) = \mu^*(t, x(t)) = -R^{-1} B'(t) [\gamma_1 \mathcal{M}(t) + \gamma_2 \mathcal{V}(t)] x(t), \quad (51)$$

which is the form of the controller's Nash equilibrium solution. Similarly, for the disturbance

$$\min_{v \in \mathcal{U}_M} \left\{ \bar{\gamma}_1 [x' \bar{\mathcal{M}} x + \dot{\bar{m}} + 2(Ax + B\mu^* + Dv)' \bar{\mathcal{M}} x + x' \bar{Q} x \right. \\ \left. + \mu^{*'} \bar{R} \mu^* + v' \bar{S} v] + tr(EWE'(\bar{\gamma}_1 \bar{\mathcal{M}} + \bar{\gamma}_2 \bar{\mathcal{V}})) \right. \\ \left. + \bar{\gamma}_2 [x' \bar{\mathcal{V}} x + \dot{\bar{v}} + 2(Ax + B\mu^* + Dv)' \bar{\mathcal{V}} x + 4 \bar{\mathcal{M}} EWE' \bar{\mathcal{M}}] \right\} = 0$$

which by minimization yields

$$w^*(t) = v^*(t, x(t)) = -\bar{S}^{-1} D'(t) [\bar{\gamma}_1 \bar{\mathcal{M}}(t) + \bar{\gamma}_2 \bar{\mathcal{V}}(t)] x(t). \quad (52)$$

Using this Nash equilibrium solution  $(\mu^*, v^*)$ , we can determine the Riccati equations by substitution. First

consider the mean of the control's cost function

$$\begin{aligned}
& \dot{\mathcal{M}} + A'\mathcal{M} + \mathcal{M}A + Q - [\tilde{\gamma}_1\bar{\mathcal{M}} + \tilde{\gamma}_2\bar{\mathcal{V}}]D\bar{S}^{-1}D'\mathcal{M} \\
& - \mathcal{M}D\bar{S}^{-1}D'[\tilde{\gamma}_1\bar{\mathcal{M}} + \tilde{\gamma}_2\bar{\mathcal{V}}] - [\gamma_1\mathcal{M} + \gamma_2\mathcal{M}]BR^{-1}B'\mathcal{M} \\
& - \mathcal{M}BR^{-1}B'[\gamma_1\mathcal{M} + \gamma_2\mathcal{V}] + [\gamma_1\mathcal{M} + \gamma_2\mathcal{M}]BR^{-1}B'[\gamma_1\mathcal{M} + \gamma_2\mathcal{M}] \\
& + [\tilde{\gamma}_1\bar{\mathcal{M}} + \tilde{\gamma}_2\bar{\mathcal{V}}]D\bar{S}^{-1}S\bar{S}^{-1}D'[\tilde{\gamma}_1\bar{\mathcal{M}} + \tilde{\gamma}_2\bar{\mathcal{V}}] = 0
\end{aligned} \tag{53}$$

where  $\mathcal{M}(t_f) = Q_f$ . Next we derive an expression for the variance. In a similar way use  $(\mu^*, \mathbf{v}^*)$  to give the equation

$$\begin{aligned}
& \dot{\mathcal{V}} + A'\mathcal{V} + \mathcal{V}A - [\tilde{\gamma}_1\bar{\mathcal{M}} + \tilde{\gamma}_2\bar{\mathcal{V}}]D\bar{S}^{-1}D'\mathcal{V} \\
& - \mathcal{V}D\bar{S}^{-1}D'[\tilde{\gamma}_1\bar{\mathcal{M}} + \tilde{\gamma}_2\bar{\mathcal{V}}] - [\gamma_1\mathcal{M} + \gamma_2\mathcal{V}]BR^{-1}B'\mathcal{V} \\
& - \mathcal{V}BR^{-1}B'[\gamma_1\mathcal{M} + \gamma_2\mathcal{V}] + 4\mathcal{M}EWE'\mathcal{M} = 0
\end{aligned} \tag{54}$$

with  $\mathcal{V}(t_f) = 0$ . Finally an expression for the mean of the disturbance's cost is given by

$$\begin{aligned}
& \dot{\bar{\mathcal{M}}} + A'\bar{\mathcal{M}} + \bar{\mathcal{M}}A + \bar{Q} - [\tilde{\gamma}_1\bar{\mathcal{M}} + \tilde{\gamma}_2\bar{\mathcal{V}}]D\bar{S}^{-1}D'\bar{\mathcal{M}} \\
& - \bar{\mathcal{M}}D\bar{S}^{-1}D'[\tilde{\gamma}_1\bar{\mathcal{M}} + \tilde{\gamma}_2\bar{\mathcal{V}}] - [\gamma_1\mathcal{M} + \gamma_2\mathcal{M}]BR^{-1}B'\bar{\mathcal{M}} \\
& - \bar{\mathcal{M}}BR^{-1}B'[\gamma_1\mathcal{M} + \gamma_2\mathcal{V}] + [\gamma_1\mathcal{M} + \gamma_2\mathcal{M}]BR^{-1}\bar{R}R^{-1}B'[\gamma_1\mathcal{M} + \gamma_2\mathcal{M}] \\
& + [\tilde{\gamma}_1\bar{\mathcal{M}} + \tilde{\gamma}_2\bar{\mathcal{V}}]D\bar{S}^{-1}D'[\tilde{\gamma}_1\bar{\mathcal{M}} + \tilde{\gamma}_2\bar{\mathcal{V}}] = 0
\end{aligned} \tag{55}$$

with  $\bar{\mathcal{M}}(t_f) = \bar{Q}_f$  and

$$\begin{aligned}
& \dot{\bar{\mathcal{V}}} + A'\bar{\mathcal{V}} + \bar{\mathcal{V}}A - [\tilde{\gamma}_1\bar{\mathcal{M}} + \tilde{\gamma}_2\bar{\mathcal{V}}]D\bar{S}^{-1}D'\bar{\mathcal{V}} \\
& - \bar{\mathcal{V}}D\bar{S}^{-1}D'[\tilde{\gamma}_1\bar{\mathcal{M}} + \tilde{\gamma}_2\bar{\mathcal{V}}] - [\gamma_1\mathcal{M} + \gamma_2\mathcal{V}]BR^{-1}B'\bar{\mathcal{V}} \\
& - \bar{\mathcal{V}}BR^{-1}B'[\gamma_1\mathcal{M} + \gamma_2\mathcal{V}] + 4\bar{\mathcal{M}}EWE'\bar{\mathcal{M}} = 0
\end{aligned} \tag{56}$$

where  $\bar{\mathcal{V}}(t_f) = 0$ . Notice that if these Riccati equations are satisfied then we know the strategy  $(\mu^*, \mathbf{v}^*)$  given in (52) and (51). This leads to the following theorem.

*Theorem 5:* Consider the stochastic game in which the system is linear and the costs are quadratic. Suppose the  $\mathcal{M}(t), \mathcal{V}(t), \mathcal{P}(t)$  are unique solutions to the coupled Riccati equations (53), (54), (55), (56), then the Nash equilibrium solution  $(\mu^*(t, x), \mathbf{v}^*(t, x))$  is given by (51) and (52).  $M(t, x), V(t, x), \bar{\mathcal{M}}(t, x)$ , and  $\bar{\mathcal{V}}(t, x)$  are then

constructed with the aid of

$$\dot{m}(t) = -tr(E(t)W(t)E'(t)\mathcal{M}(t))$$

$$\dot{v}(t) = -tr(E(t)W(t)E'(t)\mathcal{V}(t))$$

$$\dot{\bar{m}}(t) = -tr(E(t)W(t)E'(t)\bar{\mathcal{M}}(t))$$

$$\dot{\bar{v}}(t) = -tr(E(t)W(t)E'(t)\bar{\mathcal{V}}(t))$$

where  $m(t_f) = 0, v(t_f) = 0, \bar{m}(t_f) = 0, \bar{v}(t_f) = 0$ .

### B. Zero Sum Case

The system in the zero sum game will be the same as in the non-zero sum game. It will be a linear system described by (50). Along with the linear system, a quadratic cost function

$$J = \int_{t_0}^{t_f} (x'(t)Q(t)x(t) + u'(t)R(t)u(t) + w'(t)S(t)w(t))dt + x'(t_f)Q_f x(t_f)$$

will be used where  $Q_f = \bar{Q}_f = 0$ ,  $Q$  are positive semidefinite and  $R, \bar{S}$  are positive definite. Assume  $M(t, x) = x' \mathcal{M}(t)x + m(t)$ , and  $V(t, x) = x' \mathcal{V}(t)x + v(t)$ .

Using (41), we have

$$\begin{aligned} \min_{\mu \in \mathcal{U}_M} \max_{v \in \mathcal{V}_M} \left\{ \gamma_1 [x' \dot{\mathcal{M}}x + \dot{m} + 2(Ax + B\mu + Dv^*)' \mathcal{M}x + x' Qx \right. \\ \left. + \mu' R\mu + v' S v] + tr(EWE'(t)(\gamma_1 \mathcal{M} + \gamma_2 \mathcal{V})) \right. \\ \left. + \gamma_2 [x' \dot{\mathcal{V}}x + \dot{v} + 2(Ax + B\mu + Dv^*)' \mathcal{V}x + 4\mathcal{M}EWE' \mathcal{M}] \right\} = 0 \end{aligned}$$

and minimizing this gives

$$u^*(t) = \mu^*(t, x(t)) = -R^{-1}B'(t)[\gamma_1 \mathcal{M}(t) + \gamma_2 \mathcal{V}(t)]x(t) \quad (57)$$

which is the form of the controller's zero sum equilibrium solution. A similar step holds for the disturbance, which by maximization yields

$$w^*(t) = v^*(t, x(t)) = -S^{-1}D'(t)[\gamma_1 \mathcal{M}(t) + \gamma_2 \mathcal{V}(t)]x(t). \quad (58)$$

Using  $(\mu^*, v^*)$ , the matrix weight from the admissible mean cost function may be determined through substitution,

$$\begin{aligned} \dot{\mathcal{M}} + A' \mathcal{M} + \mathcal{M}A + Q - (2\gamma_1 - \gamma_1^2) \mathcal{M} [DS^{-1}D' + BR^{-1}B'] \mathcal{M} \\ - (\gamma_2 - \gamma_1 \gamma_2) \mathcal{V} [DS^{-1}D' + BR^{-1}B'] \mathcal{M} - (\gamma_2 - \gamma_1 \gamma_2) \mathcal{M} [DS^{-1}D' + BR^{-1}B'] \mathcal{V} \\ + \gamma_2^2 \mathcal{V} [DS^{-1}D' + BR^{-1}B'] \mathcal{V} = 0 \end{aligned} \quad (59)$$

where  $\mathcal{M}(t_f) = Q_f$ . Likewise for the variance, we obtain

$$\begin{aligned} & \dot{\mathcal{V}} + A'\mathcal{V} + \mathcal{V}A - \gamma_1 \mathcal{M}[BR^{-1}B + DS^{-1}D]\mathcal{V} \\ & - \gamma_1 \mathcal{V}[BR^{-1}B + DS^{-1}D]\mathcal{M} - 2\gamma_2 \mathcal{V}[BR^{-1}B + DS^{-1}D]\mathcal{V} \\ & + 4\mathcal{M}EWE'\mathcal{M} = 0 \end{aligned} \quad (60)$$

with  $\mathcal{V}(t_f) = 0$ . Notice that if these Riccati equations are satisfied then we know the strategy  $(\mu^*, v^*)$  given in (58) and (57). This leads to the following theorem.

*Theorem 6:* Consider the stochastic game in which the system is linear and the costs are quadratic. Suppose the  $\mathcal{M}(t), \mathcal{V}(t), \mathcal{P}(t)$  are unique solutions to the coupled Riccati equations (59), (60), then the zero sum equilibrium solution  $(\mu^*(t, x), v^*(t, x))$  is given by (57) and (58).  $m(t, x)$  and  $v(t, x)$  are then constructed with the aid of  $\mathcal{M}$  and  $\mathcal{V}$ , that is,

$$\begin{aligned} \dot{m}(t) &= -tr(E(t)W(t)E'(t)\mathcal{M}(t)) \\ \dot{v}(t) &= -tr(E(t)W(t)E'(t)\mathcal{V}(t)) \end{aligned}$$

where  $m(t_f) = 0, v(t_f) = 0$ .

## VI. $H_2/H_\infty$ AND $H_\infty$ GENERALIZATIONS

### A. A Cumulant Generalization of $H_2/H_\infty$ control

Here we will consider the linear system given in (50) with the regulated outputs

$$z_1(t) = C_1(t)x(t) + D_1(t)u(t)$$

$$z_2(t) = C_2(t)x(t) + D_2(t)u(t)$$

where  $D_1'D_1 = R_1$  is positive definite,  $C_1'C_1 = Q_1$  is positive semidefinite. The cost functions are then given as

$$\begin{aligned} J_1 &= \int_{t_0}^{t_f} z_1'(t)z_1(t)dt \\ J_2 &= \int_{t_0}^{t_f} (\delta^2 w'(t)w(t) - z_2'(t)z_2(t)) dt. \end{aligned} \quad (61)$$

Consider the two-norm of a function  $z(t)$  described by

$$\|z(t)\|_{2, [t_0, t_f]}^2 = \int_{t_0}^{t_f} E\{\|z(t)\|^2\}dt$$

where  $\|z(t)\|^2 = z'(t)z(t)$ . Furthermore, the induced norm on the system  $T_{zw}$  will be defined as

$$\|T_{zw}\|_{\infty, [t_0, t_f]} = \sup_w \frac{\|z\|_{2, [t_0, t_f]}}{\|w\|_{2, [t_0, t_f]}}$$

for all  $w \neq 0$  bounded power signals, that is one in which 2 norm of  $w$  exists.

Notice that minimizing the performance index of the disturbance then imposes a constraint on the input output properties of the disturbance  $w$  to the regulated output  $z_2$ . To see this, consider that for the performance index  $E\{J_2\} \geq 0$  we have

$$E \left\{ \int_{t_0}^{t_f} (\delta^2 w'(t)w(t) - z_2'(t)z_2(t)) dt \right\} \geq 0$$

but this is the same as

$$\int_{t_0}^{t_f} E \{ \|z_2(t)\|^2 \} dt \leq \delta^2 \int_{t_0}^{t_f} E \{ \|w(t)\|^2 \} dt.$$

For all  $w \neq 0$ , we have

$$\frac{\|z_2\|_{2, [t_0, t_f]}^2}{\|w\|_{2, [t_0, t_f]}^2} \leq \delta^2.$$

So trying to maximize the mean of  $J_2$  is equivalent to

$$\sup_w \frac{\|z_2\|_{2, [t_0, t_f]}}{\|w\|_{2, [t_0, t_f]}} \leq \delta.$$

This implies that  $\delta$  is a constraint on the  $H_\infty$  norm of the system.

Using Theorem 5 with the costs given in (61), we can see that the equilibrium solutions become

$$\begin{aligned} u^*(t) &= \mu^*(t, x(t)) = -R^{-1}B'(t)[\gamma_1 \mathcal{M}(t) + \gamma_2 \mathcal{V}(t)]x(t) \\ w^*(t) &= v^*(t, x(t)) = -\frac{1}{\delta^2}D'(t)[\bar{\gamma}_1 \bar{\mathcal{M}}(t) + \bar{\gamma}_2 \bar{\mathcal{V}}(t)]x(t). \end{aligned} \tag{62}$$

with the appropriate Riccati equations given in (53), (54), (55), and (56). It can be seen that if we let  $\gamma_1 = \bar{\gamma}_2 = 1$  and  $\gamma_2, \bar{\gamma}_1$  go to zero, then we obtain the equilibrium solution found in [22]. In this sense, the cost cumulant game has generalized the  $H_2/H_\infty$  control problem.

### B. A Cumulant Generalization of $H_\infty$ control

As was seen for the Nash game in [9], there are also some control applications for the minimax game as well. Similar to  $H_2/H_\infty$ , where the problem can be recast in terms of a game in which the players wish to minimize the mean of their specialized costs, the  $H_\infty$  problem may be a game, but in this case it will be a minimax, or zero-sum,

game. This problem was discussed in [2] for the deterministic case. The system will be the same as before, that is, described by (50). We will let  $z$  be the regulated output of the system,  $z(t) = C(t)x(t) + D(t)u(t)$ . In [2] the cost function for both players was given as

$$J(t, x; \mu, \nu) = \int_{t_0}^{t_f} (z'(t)z(t) - \delta^2 w'(t)w(t)) dt \quad (63)$$

where  $C'(t)C(t) = Q$ ,  $D'(t)D(t) = R$ ,  $C'(t)D(t) = 0$ ,  $Q$  is positive semidefinite,  $R$  is positive definite, and  $\delta$  is a positive constant. For the stochastic case, however, the mean of  $J$  will be used as the performance index, that is,  $M(t, x) = E\{J(t, x; \mu^*, \nu^*)\}$  (Later we have  $V(t, x) = Var\{J(t, x; \mu^*, \nu^*)\}$ ). With different definitions of the two norm, the results found in [2] are still valid.

To begin, we will similarly define the two norm of  $z$  as

$$\|z(\cdot)\|_{2, [t_0, t_f]}^2 = \int_{t_0}^{t_f} E\{\|z(t)\|^2\} dt.$$

Following the work of [2], let

$$\delta^* := \inf_{\mu \in \mathcal{U}} \|T_{z^w}\|_{\infty}$$

which, by use of the definition for  $\|\cdot\|_{\infty}$ , we have

$$\delta^* := \inf_{\mu \in \mathcal{U}} \sup_{\nu \in \mathcal{V}} \frac{\|z\|_{2, [t_0, t_f]}}{\|w\|_{2, [t_0, t_f]}}.$$

If  $\mu^*$  is the control law satisfying  $\delta^*$ , then

$$\|z_{\mu^*}\|_{2, [t_0, t_f]} \leq \delta^* \|w\|_{2, [t_0, t_f]}$$

for all  $\nu \in \mathcal{V}$  and where  $z_{\mu^*} = Cx + D\mu^*$ . But notice that this is equivalent to an upper bound on  $E_{tx}\{J\}$  of zero. In [2] (pp. 7) it is also noted that for a saddlepoint solution, the order of the minimization and maximization does not matter. So what we can say is that by solving this stochastic game, involving the mean, gives saddlepoint solutions in which the control's strategy is that of  $H_{\infty}$  control. Because the equilibrium strategies for linear stochastic minimax games involving the mean of a cost function are the same as those for a deterministic game with the same cost function, Theorem 4.1 of [2] still holds. That is if the smallest  $\delta$  for which the Riccati equation

$$\dot{\mathcal{M}}(t) + A'(t)\mathcal{M}(t) + \mathcal{M}(t)A(t) - \mathcal{M}(t) \left( B(t)R^{-1}(t)B'(t) - \frac{1}{\delta^2} D(t)D'(t) \right) \mathcal{M}(t) = 0$$

does not have a conjugate point, i.e. there is no finite escape time, then for  $\delta > 0$ , the Riccati equation does not have a conjugate point and the equilibrium strategy

$$u^*(t) = \mu^*(t, x(t)) = -R^{-1}(t)B'(t)\mathcal{M}(t)x(t)$$

$$w^*(t) = v^*(t, x(t)) = \frac{1}{\delta^2}(t)D'(t)\mathcal{M}(t)x(t)$$

where  $\mu^*$  is the  $H_\infty$  control law.

Notice that if more cumulants are used (in this case  $M$  and  $V$  previously given), as we have done previously in this paper for the zero sum case, it can be seen that the problem generalizes the  $H_\infty$  problem. This can be seen from the equilibrium solution that resulted from Theorem 6. This was given as

$$u^*(t) = \mu^*(t, x(t)) = -R^{-1}B'(t)[\gamma_1\mathcal{M}(t) + \gamma_2\mathcal{V}(t)]x(t) \quad (64)$$

$$w^*(t) = v^*(t, x(t)) = -\frac{1}{\delta^2}D'(t)[\gamma_1\mathcal{M}(t) + \gamma_2\mathcal{V}(t)]x(t).$$

If we let  $\gamma_1 = 1$  and let  $\gamma_2$  go to zero, the equilibrium solution and the mean Riccati equation tend towards the results from [2]. This, in turn, creates a generalization of  $H_\infty$  control through the use of cumulants.

## VII. FOUR STORY BUILDING APPLICATION

Here an example of building vibration control will be given using the generalized  $H_\infty$  (2 cumulant, minimax game control strategy). Further applications to structural control problems using generalized  $H_2/H_\infty$  can be seen in [9] and [10]. Here a four story building is examined. The four story structure, subject to a seismic disturbance, is given in [40]. The parameters of this 4 degree of freedom building are  $k = 350 \times 10^6$  N/m,  $m = 1.05 \times 10^6$  kg,  $c = 1.575 \times 10^6$  Ns/m. The stiffness and damping matrices are then

$$K = \begin{bmatrix} 4k & -2k & 0 & 0 \\ -2k & 3k & -k & 0 \\ 0 & -k & 2k & -k \\ 0 & 0 & -k & k \end{bmatrix} \quad C = \begin{bmatrix} 2c & -c & 0 & 0 \\ -c & 2c & -c & 0 \\ 0 & -c & 2c & -c \\ 0 & 0 & -c & c \end{bmatrix}$$

and the mass matrix is  $M = \text{diag}(2m, 2m, m, m)$ . With these definitions, consider the system given in (50), with

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ -M^{-1}B_{ch} \end{bmatrix} \quad E = \begin{bmatrix} 0 \\ F_w \end{bmatrix}$$

where

$$B_{ch} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad F_w = \frac{1}{m} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The system has two disturbances. One,  $\xi$ , is the ground acceleration of the earthquake; and the other,  $w$ , arises from uncertainties in the system. To account for some uncertainties in the stiffness and damping matrices, as well as some uncertainties in the control, we let

$$D = \begin{bmatrix} 0 & 0 & 0 \\ -M^{-1} & -M^{-1} & M^{-1}B_{ch} \end{bmatrix}.$$

The system also has regulated output  $z$ . These are given as

$$z_1 = 10^6 \begin{bmatrix} I \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} u.$$

Also in [40], a set of performance evaluation criteria is defined. First there are criteria that deal with the absolute acceleration of the building, which is calculated by the method given in [40]. The floor acceleration criteria are

$$J_1 = \max_{i,t} \{|a_{ai}(t)|\} \quad J_2 = \max_i \left\{ \sqrt{\frac{1}{T_f} \int_0^{T_f} a_{ai}^2(t) dt} \right\}$$

where  $i = 1, \dots, 4$  is the floor level. We also are interested in the interstory drifts,  $[d_{x1}, d_{x2}, d_{x3}, d_{x4}] = [x_1, x_2 - x_1, x_3 - x_2, x_4 - x_3]$ . The two indices

$$J_3 = \max_{i,t} \{|d_{xi}(t)|\} \quad J_4 = \max_i \left\{ \sqrt{\frac{1}{T_f} \int_0^{T_f} d_{xi}^2(t) dt} \right\}$$

help to capture the peak and rms performance of the interstory drifts. Furthermore the control rms and peak response is measured through

$$J_5 = \max_{i,t} \{|u_i(t)|\} \quad J_6 = \max_i \left\{ \sqrt{\frac{1}{T_f} \int_0^{T_f} u_i^2(t) dt} \right\}.$$

The simulation is performed using the time history from the 1940 El Centro earthquake. The second cumulant, zero sum (minimax) game is used to determine the control strategy. This method is compared with the LQG, MCV, and  $H_\infty$  methods. The system given for the LQG and MCV comparison is the same as in (50), except for the

TABLE I

## UN-PERTURBED SYSTEM SIMULATION RESULTS

	LQG	MCV	$H_\infty$	2-Cumulant Minimax
$J_1$	11.33	11.06	11.22	10.85
$J_2$	3.807	3.544	3.692	3.360
$J_3$	63.22	61.20	62.37	59.58
$J_4$	20.27	18.75	19.61	17.68
$J_5$	708.9	854.9	770.8	968.6
$J_6$	247.5	285.5	263.9	313.8

absence of the disturbance  $w$  term and its input matrix  $D$ . For LQG and MCV, the cost is simply given as

$$J = \int_{t_0}^{t_f} z'(t)z(t)dt$$

where  $z$  is given above. For the  $H_\infty$  controller, the system is the same as the second cumulant, minimax case, as is the cost function. The parameter  $\gamma$  is set to  $5 \times 10^{-15}$  and is used for both the second cumulant, minimax game and the MCV control design. For the minimax cumulant game and  $H_\infty$  control,  $\delta$  was chosen to be 7. The results of the simulation are given in Table I. Notice that for the un-perturbed system, significant reduction in  $J_1$ - $J_4$  occurs over LQG. The minimax case a 4.2%, 11.7%, 5.7%, and 12.8% reduction. Also, one can see that the minimax case also performed favorably to other control methods such as MCV and  $H_\infty$ . Meanwhile, in  $J_5$  and  $J_6$ , which help measure the control usage, an increase occurs.

A perturbed system model was also simulated. The perturbation was a change of  $-15\%$ , as can be seen in Table II, in the damping and stiffness matrices of the structure. In this case, the minimax case also fared well in comparison. One can see 5.9%, 10.0%, 6.7%, 11.3% reductions in  $J_1$ - $J_4$  using the minimax method presented over LQG, and again it does still does favorably when compared to MCV and  $H_\infty$ . Notice that  $J_5$  and  $J_6$  measure the price that you have to pay for these reductions.

Also shown in Fig. 1 and Fig. 2, are the responses of fourth floor interstory drift for the nominal system and the perturbed system. In these plots, one can see a reduction in the interstory drift by the use of control on the building when an earthquake hits.

TABLE II

PERTURBED SYSTEM SIMULATION RESULTS

	LQG	MCV	$H_\infty$	2-Cumulant Minimax
$J_1$	12.28	11.88	12.11	11.56
$J_2$	3.453	3.251	3.364	3.109
$J_3$	72.25	69.50	71.08	67.40
$J_4$	20.89	19.51	20.29	18.53
$J_5$	765.6	927.4	833.9	1055.5
$J_6$	235.2	273.6	251.7	302.8

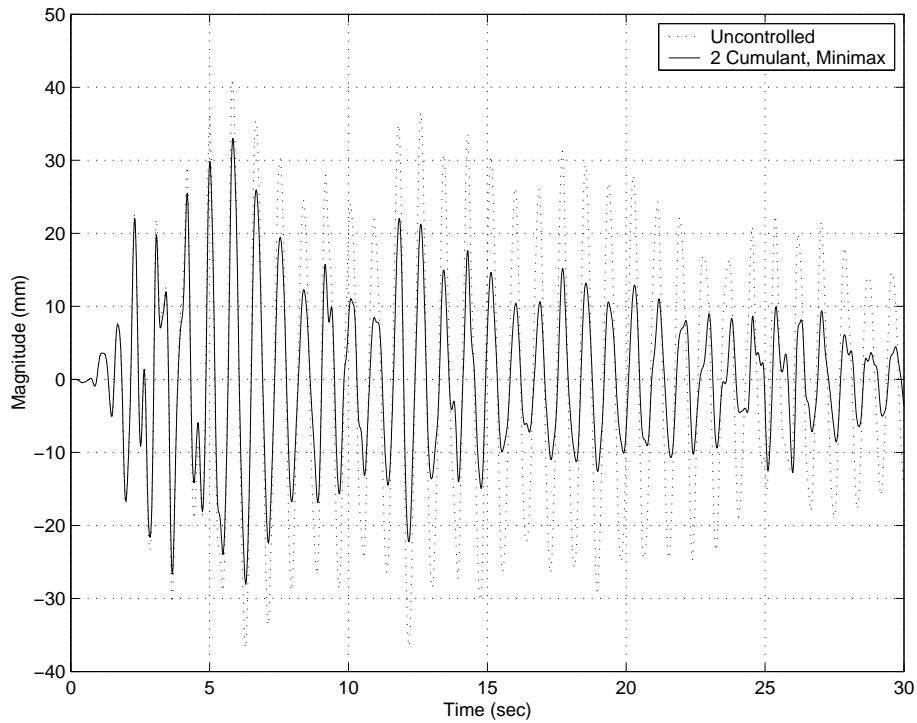


Fig. 1. Fourth Floor Interstory Drift for Nominal System

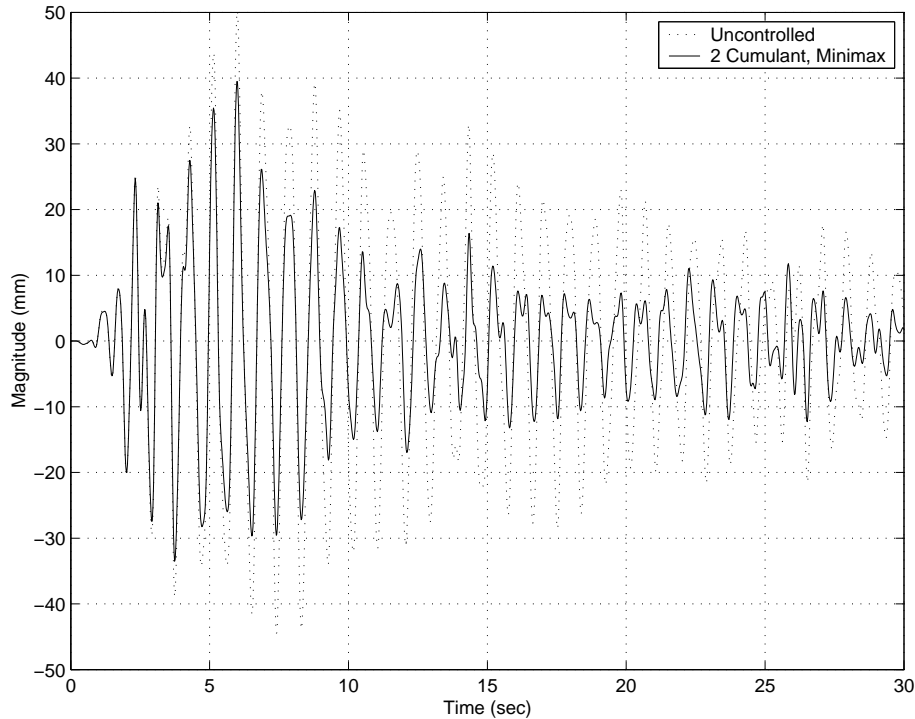


Fig. 2. Fourth Floor Interstory Drift for Perturbed System

### VIII. CONCLUSION

In this paper, we have used cumulants, in particular the first and second cumulants, in stochastic differential game theory. Minimum cost variance games were developed for both the zero and non-zero sum case. The development was for a class of nonlinear systems with non-quadratic costs. Sufficient and necessary conditions for the Nash and minimax equilibrium solutions were found, and the results were later applied to the linear quadratic case. Furthermore, a generalization of  $H_2/H_\infty$  and  $H_\infty$  control through the use of cumulants was discussed. Finally, the results were applied to a four story building control problem in which the structure was subject to a seismic disturbance.

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