Cost Moment Control and Verification Theorem for Nonlinear Stochastic Systems

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***Abstract*— In this paper we consider a system which is nonlinear in the state and linear in control action. Then we optimize the distribution of the cost function using the statistical control method. In statistical control, the optimal controller is found by optimizing any cost moments or cumulants. The optimal controller is found via the Hamilton-Jacobi-Bellman equation. As a part of statistical control, we investigate n-th moment optimal control in this paper. The Hamilton-Jacobi-Bellman equation for the n-th cost moment case is presented as a necessary condition for optimality. Then the verification theorem for n-th moment control is proved. We solve the optimizing controller for the first cost moment utilizing the derived HJB equation and the verification theorem. we verify that the new theory reduces to classical LQG result for a linear system with quadratic cost. Furthermore, we solve time-invariant nonlinear system by transforming the HJB equation to a first order partial differential equation using the pseudo-inversion method. Even though that the necessary and sufficient conditions are more easily derived, we conclude that n-th moment control generates more complicated controller then n-th cost cumulant case.**

I. Introduction

The objective of statistical control is to shape the density of the performance measure for a linear and nonlinear system in an optimal manner. In order to shape the distribution we can optimize with respect to the cumulants of distribution or the moments. In this paper, we investigate the use of cost moments as a special case of statistical control. The simplest statistical optimal control is linear-quadratic-Gaussian (LQG) control which optimizes the first moment of the performance measure. The cost variance control, which is optimization of the second cost cumulant has been studied in [7] for the linear system. More recently, the n-th cost cumulant control procedure for a quasi-linear system was presented in [8]. Continuing with the development of the statistical control, in this paper, we investigate the n-th cost moment control. We consider a system which is nonlinear in the state and linear in control action. Then we present the Hamilton-Jacobi-Bellman (HJB) equation for n-th moment control, which is a necessary condition for optimality. As the main result of this paper, we derive the verification theorem for th n-th moment optimal control, which is a sufficient condition for the optimality. Then a simple example is given. Finally, we note that cost moment control produces nonlinear controller even for a linear system with quadratic cost function.

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In the next section mathematical preliminaries are presented. Section III presents the Hamilton-Jacobi-Bellman (HJB) equation for the n-th moment case, and the verification theorem. Section IV discusses the solution procedure for the statistical control problem of the first cost moment case, and verifies that the derived HJB equations reduce to the known HJB equation in the linear system, quadratic cost case. In this section, we also derive first cost moment optimal controller for a quasi-linear system. Finally, conclusions are given in the last section.

II. Mathematical Background

In this section, we present the mathematical background necessary for statistical control. We follow the formulation given in [8].

Let $Q_0 = [t_0, t_F) \times \mathbb{R}^n$, $\bar{Q}_0$ denote the closure of Q_0, $T = [t_0, t_F]$, and let $U \subset \mathbb{R}^m$ denote a set from which control applied at any time t is chosen. Consider a nonlinear stochastic differential equation:

$$dx(t) = f(t, x(t), u(t))dt + \sigma(t, x(t))dw(t), \ , x(t_0) = x_0, \quad (1)$$

where $t \in T$, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in U$ is the control action, and $dw(t)$ is a Gaussian random process of dimension d with zero mean and covariance of $W(t)dt$. Assume $f : \bar{Q}_0 \times U \to \mathbb{R}^n$ is $C^1(\bar{Q}_0 \times U)$, and $\sigma : \bar{Q}_0 \to \mathbb{R}^{n\times d}$ is $C^1(\bar{Q}_0)$. We assume that

$$\left|\frac{\partial f(t,x,u)}{\partial x}\right| + \left|\frac{\partial f(t,x,u)}{\partial u}\right| \le \bar{c}, \ \left|\frac{\partial \sigma(t,x)}{\partial x}\right| \le \bar{c}, \quad (2)$$

$$\begin{aligned} |f(t,x,u)| &\le c(1+|x|+|u|), \text{ and} \\ |\sigma(t,x)| &\le c(1+|x|), \end{aligned} \quad (3)$$

for $(t,x) \in \bar{Q}_0$, $(t,x,u) \in \bar{Q}_0 \times U$, and constants c and $\bar{c}$. The matrix norm notation $|f|$ denotes an operator norm.

Remark. Equation (2) is a Lipschitz condition, that is a smoothness condition which most practical stochastic equations satisfy. Equation (3) is a linear growth condition which is harder to satisfy, however, if this condition is violated then it does not mean that no solution exists, but it means that the solution could go to infinity [6, p. 94]. Also the conditions (2) and (3) are sufficient conditions for unique existence of the solution $x(t)$.

Remark. For an autonomous stochastic differential equation, Lipschitz condition (2) is sufficient for the existence and uniqueness [1, p. 113].

A memoryless feedback *control law* is introduced as

$$u(t) = k(t, x(t)), \quad t \in T, \quad (4)$$

where k is a nonrandom function with random arguments. Now we admit only the bounded, Borel measurable feedback control law, $k(t,x) : \bar{Q}_0 \to U$ such that $k(t,x)$ satisfies a local Lipschitz condition and the linear growth condition. A feedback control law that satisfies both of these conditions is called *admissible*. Then a pathwise unique solution process $x(t)$ of (1) exists in probability, see [5, p. 159].

Consider a non-quadratic cost-to-go function,

$$J(t,x(t),k) = \int_t^{t_F} \Big[L(s,x(s),k(s,x(s))) \Big] ds + \psi(x(t_F)), \tag{5}$$

where $L : \bar{Q}_0 \times U \to \mathbb{R}^+$ is $C(\bar{Q}_0 \times U)$ and $\psi : \mathbb{R}^n \to \mathbb{R}^+$ is $C(\mathbb{R}^n)$. Also assume that L and ψ satisfy polynomial growth conditions. Fleming and Rishel show that a process $x(t)$ from (1), having an admissible controller k, together with polynomial growth conditions ensure that $E\{J(t,x(t),k)|x(t)=x\}$ is finite, see [4, p. 157].

Moments are defined as

$$M_i(t,x;k) = E\left\{ J^i(t,x,k) | x(t) = x \right\}.$$

From here on, we will use the shortened notation E_{tx} for the above conditional expectation. Furthermore, we introduce another notation; $\mathcal{O}(k)$ is the backward evolution operator given by

$$\mathcal{O}(k) = \frac{\partial}{\partial t} + f'(t,x,k(t,x)) \frac{\partial}{\partial x} + \frac{1}{2} \text{ tr } \left(\sigma W(t) \sigma' \frac{\partial^2}{\partial x^2} \right). \tag{6}$$

For all $(t,x) \in \bar{Q}_0$, a real valued function $\Phi(t,x)$ on $T \times \mathbb{R}^n$ satisfies a polynomial growth condition, if there exist constants k_1 and k_2 such that

$$|\Phi(t,x)| \le k_1 (1 + |x|^{k_2}). \tag{7}$$

Let $C^{1,2}(\bar{Q}_0)$ denote the space of $\Phi(t,x)$ such that Φ and the partial derivatives $\Phi_t, \Phi_{x_i}, \Phi_{x_i x_j}$ for $i,j = 1,\cdots,n$ are continuous on $\bar{Q}_0$. Also let $C_p^{1,2}(\bar{Q}_0)$ denote the space of $\Phi(t,x) \in C^{1,2}(\bar{Q}_0)$ such that $\Phi, \Phi_t, \Phi_{x_i}, \Phi_{x_i x_j}$ for $i,j = 1,\cdots,n$ satisfy a polynomial growth condition. Assumptions $\Phi(t,x) \in C_p^{1,2}(\bar{Q}_0)$, k admissible, and $E\{|x(s)|^m | x(t) = x\}$ bounded for $m = 1,2,\cdots$ and $t \le s \le t_F$ insure that the terms in the right hand side of (8), following are finite. Thus, we obtain the Dynkin formula (see [5, pages 128,135,161]),

$$\Phi(t,x) = E_{tx} \left\{ \int_t^{t_F} -\mathcal{O}(k)\Phi(s,x(s))\, ds + \Phi(t_F, x(t_F)) \right\}. \tag{8}$$

Now we define a moment around an arbitrary point, a, using Stieltjes integral.

$$M_i = \int_{-\infty}^{\infty} (x-a)^i dF$$

where $i = 0,1,2,\cdots$, $M_0 = 1$ by definition, and F is a frequency distribution function. Moments are a set of descriptive constants of a distribution that are useful for measuring its properties.

The following definitions are used to define the moment control problem.

Definition A minimal moment control law k^* satisfies

$$M_i(t,x,k^*) = M_i^*(t,x) \le M_i(t,x,k), \tag{9}$$

for $t \in T$, $x \in \mathbb{R}^n$.

A statistical control problem seeks a control law which minimizes the any cost moment or cumulant.

Remark. Here we assume the value functions are twice continuously differentiable. It is well known that the values functions are not necessarily differentiable, but as an initial investigation of the nonlinear statistical control theory, we will make the differentiability assumptions. We will investigate the use of viscosity solutions in the future research for nondifferentiable value functions.

III. n-TH MOMENT HJB EQUAITON AND VERIFICATION THEOREM

We present the necessary condition for the optimality, which was presented in [8]. Then we present the sufficient condition for optimality.

Theorem 3.1: Assume $M_i^*(t,x) \in C_p^{1,2}(\bar{Q}_0)$ and the existence of an optimal controller k^*, where $i = 1,2,3,\cdots$. By definition, M_0 is an identity. Then k^* and $M_i^*(t,x)$ satisfy the partial differential equation

$$\mathcal{O}(k^*)[M_i^*(t,x)] + iM_{i-1}^*(t,x)L(t,x,k^*(t,x)) = 0 \tag{10}$$

for $t \in T$, $x \in \mathbb{R}^n$, where

$$\begin{aligned} &\mathcal{O}(k^*)[M_i^*(t,x)] + iM_{i-1}^*(t,x)L(t,x,k^*(t,x)) \\ &\quad = \min_k \left\{ \mathcal{O}(k)[M_i^*(t,x)] + iM_{i-1}^*(t,x)L(t,x,k(t,x)) \right\}, \end{aligned} \tag{11}$$

along with the boundary condition

$$M_i^*(t_F,x) = \psi^i(x(t_F)), \quad i = 1,2,3,\cdots, \; x \in \mathbb{R}^n. \tag{12}$$

Proof. Omitted for brevity.

Now we derive a verification theorem for n-th Moment Optimization. This verification theorem states that if there exists a sufficiently smooth solution of the HJB equation, then it is the optimal cost, and using this solution an optimal control can be determined. Here we note that the moments, M_i^* are positive because we assumed the cost function in (5) is positive.

Theorem 3.2: (Verification Theorem). Let $M_i^* \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be a solution to the partial differential equation

$$0 = \min_k \left\{ \mathcal{O}(k)[M_i^*(t,x)] + iM_{i-1}^*(t,x)L(t,x,k(t,x)) \right\}, \tag{13}$$

for all $(t,x) \in Q$ with the boundary condition $M_i^*(t_F,x) = \psi^i(x)$. Then

$$M_i^*(t,x) \le M_i(t,x;k)$$

for every k and any $(t,x) \in Q$. If in addition such a k also satisfies the equation

$$\begin{aligned} &\mathcal{O}(k)[M_i^*(t,x)] + iM_{i-1}^*(t,x)L(t,x,k(t,x)) = \\ &\quad \min_{\tilde{k}} \left\{ \mathcal{O}(\tilde{k})[M_i^*(t,x)] + iM_{i-1}(t,x)L(t,x,\tilde{k}(t,x)) \right\} \end{aligned}$$

for all $(t,x)\in Q$, then $M_i^*(t,x)=M_i(t,x;k)$ and k^* is an optimal control law.

Proof. From (13) for each k and $(t,x)\in Q$

$$iM_{i-1}^*(t,x)L(t,x,k(t,x))+\mathcal{O}(k)[M_i^*(t,x)]\geq 0. \quad (14)$$

Because x is a Markov diffusion process, and $M_i^*\in C_p^{1,2}(Q)$, we may use the Dynkin formula (see [5], [7]). From the boundary condition and the Dynkin formula we obtain

$$M_i^*(t,x)=E_{tx}\left\{\int_t^{t_F}-\mathcal{O}(k)[M_i^*(s,x(s))]\,ds+\psi^i(x(t_F))\right\}. \quad (15)$$

Let $L_s = L(s,x(s),k(s,x(s)))$ and $L_r = L(r,x(r),k(r,x(r)))$, and substitute from (14) expression $-\mathcal{O}(k)[M_i^*(s,x(s))]$ in (15). Then we obtain

$$\begin{aligned}
&M_i^*(t,x)\\
&\quad\leq E_{tx}\left\{\int_t^{t_F} iL_sM_{i-1}^*(s,x(s))\,ds+\psi^i(x(t_F))\right\}\\
&\quad+E_{tx}\left\{\int_t^{t_F} iL_sE_{sx}\left\{\left[\int_s^{t_F} L_r\,dr+\psi(x(t_F))\right]\,ds\right\}^{i-1}\right.\\
&\qquad\left.+\psi^i(x(t_F))\right\}\\
&=\int_t^{t_F} E_{tx}\left\{E_{sx}\left\{iL_s\left[\int_s^{t_F} L_r dr+\psi(x(t_F))\right]^{i-1}\right\}\right\}ds\\
&\qquad+E_{tx}\left\{\psi^i(x(t_F))\right\}.
\end{aligned}$$

For a justification of interchange of the integral and the expectation in the last equation, see [3, page 62]. Now let $\mathcal{F}_t^+$ denote the smallest σ-algebra with respect to which $x(r)$ is measurable for all $r\geq t$. Suppose a random variable Z is $\mathcal{F}_t^+$–measurable and $t<s$, then

$$E_{tx}\{E_{sx}\{Z\}\}=E_{tx}\{Z\}, \quad (16)$$

which is a special form of the Chapman-Kolmogorov equation as Wong and Hajek showed in [10, page 66]. Consequently, we have

$$\begin{aligned}
&M_i^*(t,x)\leq\\
&E_{tx}\left\{\int_t^{t_F}\left[iL_s\int_s^{t_F} L_r dr+\psi(x(t_F))\right]^{i-1}\,ds\right.\\
&\left.+\psi^i(x(t_F))\right\} \quad (17)
\end{aligned}$$

Now consider the first term inside the expectation on the right side of the above equation. Swithing the limits of the integrations and using an algebraic identity, we have

$$\begin{aligned}
&\int_{t_F}^t iL_s\left[\int_{t_F}^s L_r\,dr+\psi(x(t_F))\right]^{i-1}ds\\
&=\int_{t_F}^t\left[i\sum_{k=0}^{i-1}\frac{(i-1)!}{k!(i-1-k)!}\left(\int_{t_F}^s L_r dr\right)^k\right]\\
&L_s\psi(x(t_F))^{i-1-k}ds. \quad (18)
\end{aligned}$$

Let u be the bracketed expression in the right side of the above equation, $dv=L_s\psi(x(t_F))^{i-1-k}ds$, and then use the integration by parts formula, $\int u dv = uv-\int v du$. Suppressing the argument in $\psi(x(t_F))$, we obtain

$$\begin{aligned}
&\int_{t_F}^t iL_s\left[\int_{t_F}^s L_r\,dr+\psi(x(t_F))\right]^{i-1}ds\\
&=i\sum_{k=0}^{i-1}\frac{(i-1)!}{k!(i-1-k)!}\left(\int_{t_F}^t L_r dr\right)^{k+1}\psi^{i-1-k}\\
&-\int_{t_F}^t i\sum_{k=0}^{i-1}\frac{(i-1)!k}{k!(i-1-k)!}\left(\int_{t_F}^s L_r dr\right)^k L_s\psi^{i-1-k}ds.
\end{aligned}$$

The last term in the right side can be combined with the left side term. With a little algebraic manipulation we have

$$\begin{aligned}
&\int_{t_F}^t i\sum_{k=0}^{i-1}\frac{(i-1)!(k+1)}{k!(i-1-k)!}\left(\int_{t_F}^s L_r dr\right)^k L_s\psi^{i-1-k}ds\\
&=i\sum_{k=0}^{i-1}\frac{(i-1)!}{k!(i-1-k)!}\left(\int_{t_F}^t L_r dr\right)^{k+1}\psi^{i-1-k}.
\end{aligned}$$

Divide both sides by $k+1$ and comparing with Equation (18), we obtain

$$\begin{aligned}
&\int_{t_F}^t iL_s\left[\int_{t_F}^s L_r\,dr+\psi\right]^{i-1}ds\\
&=i\sum_{k=0}^{i-1}\frac{(i-1)!}{k!(i-1-k)!}\left(\int_{t_F}^t L_r dr\right)^{k+1}\psi^{i-1-k}.
\end{aligned}$$

Let $\hat{k}=k+1$, then

$$\begin{aligned}
&\int_{t_F}^t iL_s\left[\int_{t_F}^s L_r\,dr+\psi\right]^{i-1}ds\\
&=\sum_{\hat{k}=1}^{i}\frac{i!}{\hat{k}!(i-\hat{k})!}\left(\int_{t_F}^t L_r dr\right)^{\hat{k}}\psi^{i-\hat{k}}.
\end{aligned}$$

Now the terms inside the expectation in Equation (17) can be rewritten. By reversing the limits of the integration, we have

$$\begin{aligned}
&\int_t^{t_F} iL_s\left[\int_s^{t_F} L_r\,dr+\psi(x(t_F))\right]^{i-1}ds+\psi^i(x(t_F))\\
&=\sum_{\hat{k}=0}^{i}\frac{i!}{\hat{k}!(i-\hat{k})!}\left(\int_{t_F}^t L_r dr\right)^{\hat{k}}\psi^{i-\hat{k}}\\
&=\left[\int_s^{t_F} L_r\,dr+\psi(x(t_F))\right]^i.
\end{aligned}$$

Note that the limits of the summation changed in the first equality in the above expression. Therefore, Equation (17) becomes

$$M_i^*(t,x)\leq E_{tx}\left\{\left[\int_s^{t_F} L_r\,dr+\psi(x(t_F))\right]^i\right\}.$$

Thus $M_i^*(t,x)\leq M_i(t,x;k)$. This proves the first part. For the second part, the inequality becomes equality. □

IV. First Cost Moment Optimization of a Nonlinear Stochastic System

We derive the full-state-feedback solution of the optimal statistical control problem for the first cost moment case when the system is nonlinear with respect to the state and affine linear with respect to control. The problem formulation starts from Eqs. (1) and (5), we make a few assumptions: $L(t,x,k(t,x)) = l(t,x) + k'(t,x)R(t,x)k(t,x)$, $\psi(x(t_F)) = 0$, and $f(t,x,k(t,x)) = g(t,x) + B(t,x)k(t,x)$, where k is an admissible (*i.e.*, satisfies local Lipschitz condition and linear growth condition) feedback control law; $l : \bar{Q}_0 \to \mathbb{R}^+$ is $C(\bar{Q}_0)$ and satisfies the polynomial growth conditions assumed for L; and $g : \bar{Q}_0 \to \mathbb{R}^n$ is $C^1(\bar{Q}_0)$ and satisfies the linear growth condition and the local Lipschitz condition assumed for f. Moreover $R(t,x) > 0$, and $B(t,x)$ are continuous real matrices of appropriate dimensions for all $t \in T$. Thus, the state equation that we are considering is

$$dx(t) = g(t,x)dt + B(t,x)k(t,x)dt + \sigma(t,x)dw(t) \quad (19)$$

and the cost function is

$$J(t,x,k) = \int_t^{t_F} l(t,x) + k'(t,x)R(t,x)k(t,x)dt \quad (20)$$

where $E\{dw(t)dw'(t)\} = W(t)dt$.

Here we find the controller, k, that will minimize the value function,

$$M_1^*(t,x) = \inf_k \left[E_{tx}\{J(t,x,k)\}\right] \quad (21)$$

where E_{tx} denotes the following conditional expectation, $E\{\cdot|x(t)=x\}$. With the fact that zeroth moment is defined to be an identity. We have the following partial differential equation as the necessary condition for optimality,

$$\begin{aligned}&\frac{\partial M_1^*(t,x)}{\partial t} + g'\frac{\partial M_1^*(t,x)}{\partial x} + k'(t,x)B'(t,x)\frac{\partial M_1^*(t,x)}{\partial x}\\&+\frac{1}{2}tr\left(\sigma(t,x)W\sigma'(t,x)\frac{\partial^2 M_1^*(t,x)}{\partial x^2}\right)\\&+l + k'(t,x)R(t,x)k(t,x) = 0. \quad (22)\end{aligned}$$

The minimizing controller is obtained as

$$k^*(t,x) = -\frac{1}{2}R^{-1}(t,x)B'(t,x)\frac{\partial M_1^*(t,x)}{\partial x}. \quad (23)$$

The second order necessary condition, $R(t,x) > 0$, is satisfied also. Therefore, the minimum is guaranteed, and the controller (23) is a candidate for an optimal statistical control for the first cost cumulant case.

Now, we find the partial differential equation from the necessary conditions of optimality, Eq. (22). To find the partial differential equation, we substitute the optimal controller, Eq. (23), into the Eq. (22). Simplifying the expression, we have

$$\begin{aligned}0 =& \frac{\partial M_1^*(t,x)}{\partial t} + g'(t,x)\frac{\partial M_1^*(t,x)}{\partial x}\\&-\frac{1}{4}\left(\frac{\partial M_1^*(t,x)}{\partial x}\right)' B(t,x)R^{-1}(t,x)B'(t,x)\frac{\partial M_1^*(t,x)}{\partial x}\\&+\frac{1}{2}tr\left(\sigma(t,x)W(t)\sigma'(t,x)\frac{\partial^2 M_1^*(t,x)}{\partial x^2}\right) + l(t,x). \quad (24)\end{aligned}$$

This is the necessary condition for the first moment optimization problem.

In this subsection, we find a general solution for the nonlinear time varying system.

Theorem 4.1: Assume $M_1^* \in C_p^{1,2}(\bar{Q}_0)$ and $l(t,x) = x'Q(t,x)x$ and $g(t,x) = A(t,x)x$. Furthermore assume that $M_1^*(t,x)$ is symmetric nonnegative definite matrices in the following quadratic form.

$$M_1^*(t,x) = x'\mathcal{M}_1(t,x)x + m_1(t,x). \quad (25)$$

The optimal controller that optimizes the value function, (21) is given by

$$\begin{aligned}k^*(t,x) =& -\frac{1}{2}R^{-1}(t,x)B'(t,x)\\&\left[2\mathcal{M}_1(t,x)x + vec\left\{x'\frac{\partial \mathcal{M}_1(t,x)}{\partial x_i}x\right\}\right] \quad (26)\end{aligned}$$

where $\mathcal{M}_1$ must satisfy the following partial differential equations with the boundary conditions $\mathcal{M}_1(t_F,x) = 0$ and $m_1(t_F,x) = 0$.

$$\begin{aligned}0 =& \frac{\partial \mathcal{M}_1(t,x)}{\partial t} + Q(t,x)\\&-\mathcal{M}_1(t,x)B(t,x)R^{-1}(t,x)B'(t,x)\mathcal{M}_1(t,x)\\&+A'(t,x)\mathcal{M}_1(t,x) + \mathcal{M}_1(t,x)A(t,x),\\ \frac{\partial m_1(t,x)}{\partial t} =& -tr(\sigma(t,x)W(t)\sigma'(t,x)\mathcal{M}_1(t,x)), \quad (27)\end{aligned}$$

and surpressing the arguments

$$\begin{aligned}0 =& -\frac{1}{2}vec\left\{x'\frac{\partial \mathcal{M}_1}{\partial x_i}x\right\}' BR^{-1}B'\mathcal{M}_1 x\\&-\frac{1}{2}x'\mathcal{M}_1 B(t,x)R^{-1}B' vec\left\{x'\frac{\partial \mathcal{M}_1}{\partial x_i}x\right\}\\&-\frac{1}{4}vec\left\{x'\frac{\partial \mathcal{M}_1}{\partial x_i}x\right\}' BR^{-1}B' vec\left\{x'\frac{\partial \mathcal{M}_1}{\partial x_i}x\right\}\\&+tr\left(\sigma W(t)\sigma' vec\left\{x'\frac{\partial \mathcal{M}_1}{\partial x_i}'\right\}\right)\\&+tr\left(\sigma W(t)\sigma' vec\left\{\frac{\partial \mathcal{M}_1}{\partial x_i}x\right\}\right)\\&+\frac{1}{2}tr\left(\sigma W(t)\sigma' vec\left\{x'\frac{\partial^2 \mathcal{M}_1}{\partial x_i^2}x\right\}\right)\\&+x'A' vec\left\{x'\frac{\partial \mathcal{M}_1}{\partial x_i}x\right\}. \quad (28)\end{aligned}$$

Proof. Omitted for brevity.

The necessary conditions given by Equation (28) is difficult to satisfy for a given $A(t,x)$ in multivariable case. Thus, as the next step, we will consider two special cases–a linear system with a quadratic cost function and time invariant system–to present the complete solution of the first cost moment optimization problem.

Now, we verify that the developed theory for the n-th moment case provides the solutions to the linear system and quadratic cost function as expected. Thus, we assume a linear system and a quadratic cost function. We verify that this solution is equivalent to the classical LQG case.

Corollary 4.1: Assume M_1^* is smooth. Also assume that $l(t,x) = x'Q(t)x$, $R(t,x) = R(t)$, $g(t,x) = A(t)x$, $B(t,x) = B(t)$, and $\sigma(t,x) = E(t)$. We assume a quadratic form as

$$M_1^*(t,x) = x'\mathcal{M}_1(t)x + m_1(t). \tag{29}$$

The optimal controller that optimizes the value function, (21) is given by

$$k^*(t,x) \quad = \quad -R^{-1}(t)B'(t)\mathcal{M}_1(t)x,$$

where $\mathcal{M}_1$ is a solution of the following partial differential equations.

$$\begin{aligned} 0 \quad = \quad & \dot{\mathcal{M}}_1(t) + Q(t) - \mathcal{M}_1(t)B(t)R^{-1}(t)B'(t)\mathcal{M}_1(t) \\ & + A'(t)\mathcal{M}_1(t) + \mathcal{M}_1(t)A(t), \end{aligned} \tag{30}$$

with the boundary condition $\mathcal{M}_1(t_F) = 0$. And the following equation:

$$\dot{m}_1(t) \quad = \quad -tr(E(t)W(t)E'(t)\mathcal{M}_1(t)), \tag{31}$$

with the boundary condition $m_1(t_F) = 0$.

Proof. Omitted for brevity.

Now, we solve the first moment control problrm for a non-linear time invariant sytem using pseudo-invresion method. Consider

$$dx(t) = g(x)dt + B(x)k(x)dt + \sigma(x)dw(t)$$

with the cost function,

$$J(x,k) = \int_t^{t_F} [l(x) + k'(x)R(x)k(x)]dt.$$

Then we optimize the value function,

$$M_1^*(x) = \inf_k (E_{tx}\{J(x,k)\}).$$

We also have the following HJB equation for the time invariant system,

$$\begin{aligned} 0 \quad = \quad & g'(x)\frac{\partial M_1^*(x)}{\partial x} - \frac{1}{4}\left(\frac{\partial M_1^*(x)}{\partial x}\right)' B(x) \\ & R^{-1}(x)B'(x)\frac{\partial M_1^*(x)}{\partial x} \\ & + \frac{1}{2}tr\left(\sigma(x)W(t)\sigma'(x)\frac{\partial^2 M_1^*(x)}{\partial x^2}\right) + l(x). \end{aligned} \tag{32}$$

The optimal controller is given by

$$k^*(x) = -R^{-1}(x)B'(x)\frac{\partial M_1^*(x)}{\partial x}. \tag{33}$$

Theorem 4.2: Let $P(x) = B(x)R^{-1}(x)B'(x)$. For the HJB equation (32), we assume that there exists a function $\rho(x) : \mathbb{R}^n \to \mathbb{R}^n$, such that

$$\begin{aligned} 0 \quad = \quad & -\rho'(x)P^+(x)\rho(x) + g'(x)P^+(x)g(x) + l(x) \\ & + tr\Bigg[\sigma(x)W\sigma'(x)P^+(x)\left(\frac{\partial g(x)}{\partial x'} \pm \frac{\partial \rho(x)}{\partial x'}\right) \\ & + \sigma(x)W\sigma'(x)\frac{\partial P^+(x)}{\partial x'}(I_n \otimes (g(x) \pm \rho(x)))\Bigg]. \end{aligned} \tag{34}$$

where the matrix P^+ is the pseudo-inverse of P. Then

$$\frac{\partial M_1^*(x)}{\partial x} \quad = \quad 2P^+(x)(g(x) \pm \rho(x)), \tag{35}$$

and the optimal controller is given by

$$k^*(x) = -R^{-1}(x)B'(x)P^+(x)(g(x) \pm \rho(x)). \tag{36}$$

Proof. Omitted for brevity.

Remark. Equation (24) is a second order nonlinear partial differential equation. Here we call Eq. (24) the stochastic Hamilton-Jacobi-Bellman equation, which is also known in the literature as the dynamic programming equation [4], the stochastic Bellman equation [2], and the Hamilton-Jacobi-Bellman (HJB) equation [5]. Eq. (24) without the trace term is also known as the HJB equation. Here, we distinguish between those two by calling Eq. (24) as the stochastic HJB equation and the one without the trace term as the deterministic HJB equation. In Game theory, the partial differential equation that is analogous to HJB equation is known as the Hamilton-Jacobi-Issacs (HJI) equation.

Computational Details. We have changed the problem of solving a nonlinear second order partial differential equation–HJB–into a nonlinear first order partial differential equation–discriminant equation. Computationally, in order to solve the HJB equation (24) for symmetric M_1^*, we have to solve for ρ with two requirements. The first requirement is that M_1^* is symmetric and the second requirement is that M_1^* is semipositive definite. The other (symmetry) requirement is given as

$$\begin{aligned} & P^+\left(\frac{\partial g}{\partial x'} \pm \frac{\partial \rho}{\partial x'}\right) + \frac{\partial P^+}{\partial x'}(I_n \otimes (g \pm \rho)) \\ = & \left(\frac{\partial g}{\partial x'} \pm \frac{\partial \rho}{\partial x'}\right)' P^+ + (I_n \otimes (g \pm \rho))'\left(\frac{\partial P^+}{\partial x'}\right)' \end{aligned} \tag{37}$$

and the non-negative definite requirement is given as

$$P^+\left(\frac{\partial g}{\partial x'} \pm \frac{\partial \rho}{\partial x'}\right) + \frac{\partial P^+}{\partial x'}(I_n \otimes (g \pm \rho)) \geq 0. \tag{38}$$

Example. As an example, we consider the following system

$$dx(t) = [x(t) - x^3(t)]dt + k(t, x(t))dt + dw(t)$$

with a cost function

$$J = \int_t^{t_F} \frac{1}{2}(x^2 + k^2)dt.$$

This is an autonomous system, so just the Lipschitz condition has to be satisfied, and not the growth condition. Now, we find the solutions of minimal first moment control problem, in which we find a controller such that $M_1^*(t,x) = \inf_k E_{tx}\{J\}$. In this example, we take $g(x) = x - x^3, P = BR^{-1}B' = 2$, and $\sigma = 1$. The covariance matrix of dw is W. Substituting these values into the discriminant equation (34), we obtain

$$\begin{aligned} 0 \quad = \quad & (x - x^3)\frac{1}{2}(x - x^3) - \frac{1}{2}\rho^2 + x'\frac{1}{2}x \\ & + tr\left[W\left(\frac{1}{2}(1 - 3x^2) \pm \frac{1}{2}\frac{\partial \rho}{\partial x}\right)\right]. \end{aligned}$$

This simplifies to

$$0 = x^6 - 2x^4 + 2\left(1 - \frac{3}{2}W\right)x^2 + W - \rho^2 \pm \frac{\partial \rho}{\partial x}W. \quad (39)$$

Using the method of characteristics, we have

$$\frac{dx}{W} = \frac{d\rho}{\rho^2 - \alpha(x)}$$

where $\alpha(x) = x^6 - 2x^4 + 2(1 - 3W/2)x^2 + W$. Integrating both sides, we come up with the following solution

$$\rho(x) = -\frac{\exp\left(2x\sqrt{\alpha(x)}/W + c\right) + 1}{\exp\left(2x\sqrt{\alpha(x)}/W + c\right) - 1}\sqrt{\alpha(x)} \stackrel{\text{def}}{=} -c_\rho\sqrt{\alpha(x)}, \quad (40)$$

where $\rho(x) > \sqrt{\alpha(x)}$ or $\rho(x) < -\sqrt{\alpha(x)}$, and $c \in \mathbb{R}$ is an arbitrary constant. Substituting Eq. (40) into Eq. (35) we obtain,

$$\frac{\partial M_1^*}{\partial x} = x - x^3 \mp \left[\frac{\exp(2x\sqrt{\alpha}/W + c) + 1}{\exp(2x\sqrt{\alpha}/W + c) - 1}\sqrt{\alpha}\right].$$

Consequently, the optimal controller is

$$k^* = -x + x^3 \pm \left[\frac{\exp\left(2x\sqrt{\alpha(x)}/W + c\right) + 1}{\exp\left(2x\sqrt{\alpha(x)}/W + c\right) - 1}\sqrt{\alpha(x)}\right].$$

The first requirement, Eq. (37), is satisfied because P^+ is a constant matrix and the system has a single input and a single output. The second requirement Eq. (38) is more involved and we have to verify that it is satisfied. Using Eq. (39) and $\partial g/\partial x = 1 - 3x^2$, we obtain the second requirement as

$$\left(\frac{1}{2W}c_\rho^2 - \frac{1}{2W}\right)x^6 + \left(\frac{1}{W} - \frac{1}{W}c_\rho^2\right)x^4 + \left(\frac{1}{W}c_\rho^2 - \frac{3}{2}c_\rho^2 - \frac{1}{W}\right)x^2 + \frac{1}{2}c_\rho^2 \geq 0$$

where $\rho(x) = -c_\rho\sqrt{\alpha(x)}$ and c_ρ is given in Eq. (40).

If we assume $W = 0$ for the deterministic case we obtain the following optimal controller:

$$k^* = -x + x^3 \pm \sqrt{x^6 - 2x^4 + 2x^2}.$$

Remark. If we were to perform two cost moment minimization, we will obtain a nonlinear controller even for the linear, $g = Ax$, quadratic cost case, $L(t, x, k) = x'Qx + k'Rk$. To see this, consider two partial differential equations for the first two moments from Eq. (10):

$$\begin{aligned} 0 &= \frac{\partial M_1}{\partial t} + x'Qx + k'Rk + \frac{1}{2}tr\left(\sigma W\sigma'\frac{\partial^2 M_1}{\partial x^2}\right) \\ &\quad + x'A'\frac{\partial M_1}{\partial x} + k'B'\frac{\partial M_1}{\partial x}, \\ 0 &= \frac{\partial M_2}{\partial t} + 2M_1(x'Qx + k'Rk) + \frac{1}{2}tr\left(\sigma W\sigma'\frac{\partial^2 M_2}{\partial x^2}\right) \\ &\quad + x'A'\frac{\partial M_2}{\partial x} + k'B'\frac{\partial M_2}{\partial x}. \end{aligned}$$

After multiplying a Lagrange multiplier and taking derivatives with respect to k, we find the optimal first two moment controller as

$$k^* = -\frac{1}{2 + 4\gamma M_1}R^{-1}B'\left(\frac{\partial M_1}{\partial x} + \gamma\frac{\partial M_2}{\partial x}\right).$$

If we let $M_i(t,x) = x'\mathcal{M}_i x + m_i$ for $i = 1, 2$, we obtain a nonlinear controller.

V. Conclusions

This paper presented a method to control the density of a quasi-linear system with a quasi-quadratic cost function by controlling moments of the cost function. The main results of this paper is the necessary and sufficient conditions for the n-th cost moment optimization problem. As an example, we solved the statistical control problem for first cost moment minimization case. We verified that the derived results reduces to classical LQG results. Then we solved a nonlinear time-invariant system by transforming the HJB equation into a first order partial differential equation using the pseudo-inversion method. In the last section, we showed that in the second moment optimal control, a nonlinear controller is obtained even for a linear system with quadratic cost case. Thus, we conclude that using moments may not be the best method. We propose to use cost cumulants instead.

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