Infinite-Time Minimal Cost Variance Control and Coupled Algebraic Riccati Equations

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Abstract

Minimum cost variance control (MCV) optimizes the variance of the cost function while the cost mean is kept at a prespecified level. The solutions of the infinite time horizon full-state-feedback MCV problem are found using the Hamilton-Jacobi theory. In the solutions of infinite time horizon MCV control problem, a pair of coupled algebraic Riccati equations arises. This paper considers the existence of a positive semidefinite solution pair for the steady-state version of coupled algebraic Riccati equations, where one entry of the pair corresponds to cost mean, while the other entry of the pair corresponds to cost variance. For the MCV control problem, existence and uniqueness of the solutions of the coupled algebraic Riccati equations are provided. From this result it is established that the MCV feedback controller stabilizes the closed loop system. Furthermore, the algorithm to find the MCV controller from the coupled algebraic Riccati equations is presented. Finally, three examples are provided to verify the existence theorem of the coupled algebraic Riccati equations.

1 Introduction

The idea of minimal cost variance control originated with the dissertation work of Sain [6] in 1965 at the University of Illinois. That work, which was for open-loop control, was subsequently published [7]. Liberty continued to study characteristic functions of integral quadratic forms, as well as methods for generating cumulants in the time domain. In this way, he and Hartwig further developed and expanded the minimal cost variance control idea [3]. A finite-horizon, full-state-feedback MCV control problem has been solved and reported [8]. The preliminary version of this paper was presented in [9]

The present paper builds upon [8] by going into the details of the pair of coupled Riccati equations which were introduced there. The nonlinear problem formulation is given in the next section, and linear assumptions are made in Section 2 to solve the optimal linear MCV controller. In Section 2 full-state-feedback MCV control in infinite time horizon is discussed. Section 3 addresses the existence of the steady-state solutions of Riccati equations, in the symmetric and positive semidefinite class. For MCV control, where the cost mean is kept at a prespecified level, it is also shown, in Section 3, that such solutions exist and are unique, and that they stabilize the resulting feedback system. In Section 4, infinite time horizon MCV solution algorithm is provided with three examples to verify the existence theorem of the coupled algebraic Riccati equations. Riccati equations have been used extensively to find solutions to typical problems of high performance and robust control, involving methods of the H_∞ theory and other normed optimizations, the theory of dynamic games, and risk-sensitive optimization theory. As for coupled Riccati equations, in 1998 Freiling *et al.* have independently investigated the coupled Riccati equations in [2]. The difference between their results and ours are discussed in Section 5.

Why minimal cost variance control? As pointed out by Mariton [4], the question of robustness with respect the underlining stochastic process in important. Also, it is the performance of the sample path that we should be more concerned, and minimal mean does not consider variance or the distribution of the cost function, thus it does not guarantee anything about the sample path. However, cost variance indicates what extent the performance is spread around its mean value, and it may play more important role than the mean. Two examples provided by Mariton are (1) in manufacturing, quality control is a critical profit factor, and here the ability to obtain a sharp product quality distribution curve where the mean quality can be pushed close to the rejection threshold is very important. In other words, the quality control would like to have small variance of the cost function even though that may increase the mean value. (2) In economic planning, an investor wants to maximize the expected return, however, it has to be within acceptable variance to avoid bankrupcy. Other example is in satellite attitude control, where the objective of controlling the variance of the cost is more critical than the mean because the variance is directly related to the coverage or field of view of the satellite.

2 Infinite Time Horizon Full-State-Feedback MCV Control

Full-state-feedback minimal cost variance control for the finite-time horizon was presented in [8]. Here we present the infinite-time horizon MCV control. Consider the time invariant linear system given by (1) with,

$$dx(t) = Ax + Bk(x)\,dt + E\,dw(t), \tag{1}$$

with the quadratic cost given by (2) with,

$$J(t,x,k) = \int_t^{t_F} \left[x'Qx + k'(x)Rk(x)\right] ds, \tag{2}$$

where $A, B, E, Q \geq 0$, and $R > 0$ are real constant matrices of appropriate dimensions.

We formulate the infinite time horizon state-feedback MCV optimal control problem as follows. Define

$$\begin{aligned} \bar{V}_1(x;k) &= \lim_{t_F\to\infty} \frac{1}{t_F} E\{J(t,x(t),k)\} \\ &= \lim_{t_F\to\infty} \frac{1}{t_F} V_1(t,x;k). \end{aligned} \tag{3}$$

Then we minimize the cost variance of the form,

$$\begin{aligned} \bar{V}(x;k) &= \lim_{t_F\to\infty} \frac{1}{t_F}\left[E\{J^2(t,x(t),k)\}\right. \\ &\qquad \left. -E^2\{J(t,x(t),k)\}\right] \\ &\stackrel{\text{def}}{=} \lim_{t_F\to\infty} \frac{1}{t_F}\left[V_2(t,x;k) - V_1^2(t,x;k)\right]. \end{aligned} \tag{4}$$

Note that the above represents a minimization of the cost variance per unit time. This addresses the question how much does it cost (in terms of cost variance) to operate a given system per unit time. It is also possible to formulate the problem in terms of the total cost over an infinite horizon, but this may result in infinite total cost.

Now we admit only the bounded, Borel measurable feedback control law $k(x) : \mathbb{R}^n \to U$ such that $k(x)$ satisfies a global Lipschitz condition. Moreover, we also require that $k(x)$ satisfy the linear growth condition. A feedback control law k which satisfies both of these conditions is called *admissible.* Furthermore, k is *admissible*$^\infty$ if it is admissible and $\bar{V}_1$ in (3) and $\bar{V}$ in (4) exist. Then it is known that, under these conditions (1) has a well-defined solution process, and it is a Markov diffusion process on $\mathbb{R}^n$ [5, page 1551].

Definition 1: A function $\bar{M} : \mathbb{R}^n \to \mathbb{R}^+$, which is $C^2(\mathbb{R}^n)$, is an *admissible mean cost function* if there exists an admissible$^\infty$ control law k such that $\bar{V}_1(x;k) = \bar{M}(x)$ for $x \in \mathbb{R}^n$.

Definition 2: Every admissible $\bar{M}$ defines a class $K_{\bar{M}}$ of control laws k corresponding to $\bar{M}$ in the manner that $k \in K_{\bar{M}}$ when k is an admissible control law which satisfies Definition 1.

Definition 3: Let $\bar{M}$ be an admissible cost function, and let $K_{\bar{M}}$ be its induces class of admissible control laws. An MCV control law $k^*_{\bar{V}|\bar{M}}$ such that

$$\bar{V}(x;k^*_{\bar{V}|\bar{M}}) = \bar{V}^*(x) \leq \bar{V}(x;k), \tag{5}$$

for $x \in \mathbb{R}^n$ whenever $k \in K_{\bar{M}}$.

The cost function (2) is unbounded as t_F approaches infinity. Thus typically in minimal mean case, the average cost per unit time given in Equation (3) is optimized. In infinite time horizon RS control, the cost is given by

$$J_{RS}(t,x,k) = \lim_{t_F\to\infty} \frac{1}{t_F} \ln E\left(\exp J(t,x,k)\right),$$

and the interpretation is given in terms of the large deviation theory [5]. In the similar spirit, we have formulated the infinite time horizon MCV control problem.

Assume that $\bar{M}(x) = x'\bar{\mathcal{M}}x + \bar{m}$ and $\bar{V}^*(x) = x'\bar{\mathcal{V}}x + \bar{v}$ because the cost-to-go of any cumulant of an integral quadratic cost function for a LQG problem is affine quadratic in the state [3]. We have the following results.

Lemma 2.1 *Given a gain matrix* $K(s) = -R^{-1}B'(\mathcal{M}_F(s) + \gamma\mathcal{V}_F(s))$ *such that* $A + BK(s)$ *is exponentially stable, then* $\lim_{s\to\infty}\mathcal{M}_F(s)$ *and* $\lim_{s\to\infty}\mathcal{V}_F(s)$ *exist where* $\mathcal{M}_F(s) = \mathcal{M}(s - t_F), \mathcal{V}_F(s) = \mathcal{V}(s - t_F)$,

$$\begin{aligned} -\dot{\mathcal{M}}_F(s) &= A'\mathcal{M}_F(s) + \mathcal{M}_F(s)A + Q \\ &\quad -\mathcal{M}_F(s)BR^{-1}B'\mathcal{M}_F(s) \\ &\quad +\gamma^2\mathcal{V}_F(s)BR^{-1}B'\mathcal{V}_F(s), \end{aligned} \tag{6}$$

with the boundary condition $\mathcal{M}_F(0) = 0$, *and*

$$\begin{aligned} -\dot{\mathcal{V}}_F(s) &= 4\mathcal{M}_F(s)EWE'\mathcal{M}_F(s) + A'\mathcal{V}_F(s) \\ &\quad +\mathcal{V}_F(s)A - \mathcal{M}_F(s)BR^{-1}B'\mathcal{V}_F(s) \\ &\quad -\mathcal{V}_F(s)BR^{-1}B'\mathcal{M}_F(s) \\ &\quad -2\gamma\mathcal{V}_F(s)BR^{-1}B'\mathcal{V}_F(s), \end{aligned} \tag{7}$$

with the boundary condition $\mathcal{V}_F(0) = 0$.

Following two Lemmas are used in the solution of the MCV problem and the proofs are given in [8].

Lemma 2.2 *Let* $f(x)$ *be given by*

$$f(x) = \frac{\|x\|_{\mathcal{R}}}{\|x\|_{\mathcal{V}BR^{-1}B'\mathcal{V}}} \tag{8}$$

and consider the controller term $-f(x)R^{-1}B'\mathcal{V}$. *If this term is a morphism of vector addition, then* $f(x)$ *is constant for all* x *such that* $B'\mathcal{V}x$ *is nonzero.*

Lemma 2.3 *Let $\mathcal{R}$ and $\mathcal{V}BR^{-1}B'\mathcal{V}$ have identical null spaces, and consider the function $f(x)$ defined by (8) on the domain in which $\|x\|_{\mathcal{V}BR^{-1}B'\mathcal{V}}$ does not vanish. Then $f(x)$ is equal to a (positive) constant γ on this domain, if and only if $\mathcal{R} = \gamma^2\mathcal{V}BR^{-1}B'\mathcal{V}$ on the domain.*

Denote $\bar{\mathcal{V}} = \lim_{s\to\infty}\mathcal{M}_F(s)$ and $\bar{\mathcal{M}} = \lim_{s\to\infty}\mathcal{V}_F(s)$. Furthermore, define $\check{M}(t) = t_F\mathcal{M}(t)$ and $\check{V}(t) = t_F\mathcal{V}(t)$. Then we have the following theorem.

Theorem 2.1 *In infinite time horizon, the full-state-feedback linear MCV control in $K_{\bar{M}}$ has the form*

$$k^*_{\check{V}|\check{M}}(x) = -R^{-1}B'(\bar{\mathcal{M}} + \gamma\bar{\mathcal{V}})x, \tag{9}$$

where the positive semidefinite matrices $\bar{\mathcal{M}}$ and $\bar{\mathcal{V}}$ are solutions of the coupled algebraic Riccati equations:

$$0 = A'\bar{\mathcal{M}} + \bar{\mathcal{M}}A + Q - \bar{\mathcal{M}}BR^{-1}B'\bar{\mathcal{M}} + \gamma^2\bar{\mathcal{V}}BR^{-1}B'\bar{\mathcal{V}} \tag{10}$$

and

$$\begin{aligned} 0 \;=\; & 4\bar{\mathcal{M}}EWE'\bar{\mathcal{M}} + A'\bar{\mathcal{V}} + \bar{\mathcal{V}}A - \bar{\mathcal{M}}BR^{-1}B'\bar{\mathcal{V}} \\ & -\bar{\mathcal{V}}BR^{-1}B'\bar{\mathcal{M}} - 2\gamma\bar{\mathcal{V}}BR^{-1}B'\bar{\mathcal{V}}. \end{aligned} \tag{11}$$

Theorem 2.1 states that the solution of the coupled algebraic equations (10) and (11) give the full-state-feedback linear MCV control law. However, the existence and uniqueness of the coupled algebraic Riccati equations need to be shown. In the next section this question is discussed in detail.

3 Existence and Stabilizing Properties

In this section, the existence and stabilizing properties of the coupled algebraic reccati equations are discussed. Consider the steady-state version of the pair of Riccati equations given in Equations (10) and (11). We examine those equations for solutions in the symmetric and positive semidefinite class. Under reasonable assumptions and a special condition, it is established that such a solution pair exists, and that it results in a stabilizing feedback control law for the system being controlled. Furthermore, for the MCV control problem where M is kept at a prespecified level, we do have existence and uniqueness of $\mathcal{M}^* + \gamma\mathcal{V}^*$. We begin by reviewing some of the classical results associated with Riccati equations, and we conclude the section comparing the results with the published results in [2]. The proofs of the following four claims are given in [10].

Proposition 3.1 *If Ξ_k, Q are symmetric maps such that $\Xi_k \geq Q, k = 1,2,\ldots$, and $\Xi_k \downarrow$, then $\Xi \stackrel{\triangle}{=} \lim_{k\to\infty}\Xi_k$ exists.*

Theorem 3.1

1. *If $C_1'C_1 = C_2'C_2$ and (C_1, A) is observable (respectively detectable) then (C_2, A) is observable (respectively detectable).*
2. *If $Q \geq 0$ and $(\sqrt{Q}, A)$ is observable (respectively detectable), then for all $M \geq 0$, $N > 0$ and all B, F, the pair $(\sqrt{M + Q + F'NF}, A + BF)$ is observable (respectively detectable).*

Lemma 3.1 *If $Q \geq 0$ and A is stable, the linear equation $A'\Xi + \Xi A + Q = 0$ has a unique solution Ξ, and $\Xi \geq 0$.*

Lemma 3.2 *Suppose $\Xi \geq 0$, $Q \geq 0, (\sqrt{Q}, A)$ is detectable and $A'\Xi + \Xi A + Q = 0$. Then A is stable.*

Now we are ready to state and prove the main theorem of this section.

Theorem 3.2 *Assume (A, B) is stabilizable, $(\sqrt{Q}, A)$ is detectable, $Q \geq 0$, $R > 0$, and γ is a nonnegative constant. Then the coupled algebraic Riccati equations which result from the steady-state of the pair given in Theorem 2.1 have a solution $\mathcal{M}^* + \gamma\mathcal{V}^*$, in the class of symmetric, positive semidefinite maps when the following condition is satisfied*

$$4\gamma\bar{\mathcal{M}}_k EWE'\bar{\mathcal{M}}_k + \Gamma_k - 4\gamma\bar{\mathcal{M}}_{k+1}EWE'\bar{\mathcal{M}}_{k+1} \geq 0, \tag{12}$$

where $\{K_k, \bar{\mathcal{M}}_k, \bar{\mathcal{V}}_k; k = 1,2,\ldots\}$ are constrained sequence and $\Gamma_k = (K_k - K_{k+1})'R(K_k - K_{k+1})$. Moreover, $A - BR^{-1}B'(\mathcal{M}^* + \gamma\mathcal{V}^*) = A + BK^*$ is stable, and the sequence $\mathcal{M}_k + \gamma\mathcal{V}_k$ is monotonically decreasing.

Theorem 3.2 states that if Equation (12) is satisfied then the coupled algebraic Riccati equation gives a solution, $\mathcal{M}^* + \gamma\mathcal{V}^*$, where the sequence $\mathcal{M}_k + \gamma\mathcal{V}_k$, converges to $\mathcal{M}^* + \gamma\mathcal{V}^*$. We do not have uniqueness of the solution as the classical LQG case. In fact by a simple example in the sequel, we will show that there could be multiple solutions. Thus, the solution may be a suboptimal solution if there exists other solutions. Utilizing Theorem 3.2, we have the following results for the MCV problem.

Theorem 3.3 *Under the assumptions of Theorem 3.2 and suppose that $\mathcal{M}_k = \mathcal{M}_{k+1}$, which is equal to a prespecified $\mathcal{M}$. Then $\mathcal{M}_k + \gamma\mathcal{V}_k \downarrow$ and there exist a unique solution $\mathcal{M}^* + \gamma\mathcal{V}^*$, in the class of symmetric, positive semidefinite maps. Moreover, $A - BR^{-1}B'(\mathcal{M}^* + \gamma\mathcal{V}^*) = A + BK^*$ is stable.*

The existence of the solution holds for $\bar{\mathcal{M}}_k \geq \bar{\mathcal{M}}_{k+1}$, however we need $\mathcal{M}_k = \mathcal{M}_{k+1} = \mathcal{M}$ for uniqueness. Because we specify M at a certain level for MCV control, $\mathcal{M}_k = \mathcal{M}_{k+1}$ in the MCV control problem, we have existence and uniqueness of the coupled algebraic Riccati equations. Furthermore, $A + BK^*$ is stable.

Corollary 3.1 *Under the assumptions of Theorem 3.2 and suppose $E = 0$, which is the deterministic case. Then*

$\bar{\mathcal{M}}_k + \gamma\bar{\mathcal{V}}_k \downarrow$ *and there exist a solution* $\bar{\mathcal{M}}^* + \gamma\bar{\mathcal{V}}^*$, *in the class of symmetric, positive semidefinite maps. Moreover,* $A - BR^{-1}B'(\mathcal{M}^* + \gamma\mathcal{V}^*) = A + BK^*$ *is stable.*

4 Infinite Time Horizon MCV Solution Algorithm

This section presents an algorithm to find the coupled algebraic Riccati type equations. This algorithm follows Theorem 3.2 and finds $\mathcal{M}$ and $\mathcal{V}$ when the sufficient existence condition (12) is satisfied. Even though when $\mathcal{M}$ is fixed, we have unique optimal control law, we usually do not know what $\mathcal{M}$ should be. Thus we propose the following algorithm to find all $\gamma, \mathcal{M}$, and $\mathcal{V}$. Then choose appropriate γ to find the optimal MCV control law. Given γ, the algorithm is as follows.

1. Check (A,B) stabilizable and $(\sqrt{Q},A)$ detectable.
2. Choose K_1 so that $A+BK_1$ is stable.
3. Let $i=1$.
4. Find $\bar{\mathcal{M}}_i$ from the following Lyapunov-type equation.
$$0 = (A+BK_i)'\bar{\mathcal{M}}_i + \bar{\mathcal{M}}_i(A+BK_i) + Q + K_i'RK_i$$
5. Find $\bar{\mathcal{V}}_i$ from the following Lyapunov-type equation.
$$0 = (A+BK_i)'\bar{\mathcal{V}}_i + \bar{\mathcal{V}}_i(A+BK_i) + 4\bar{\mathcal{M}}_iEWE'\bar{\mathcal{M}}_i$$
6. Let $K_{k+1} = -R^{-1}B'(\bar{\mathcal{M}}_k + \gamma\check{\mathcal{V}}_k)$ and find K_{i+1}.
7. If $i<2$, let $i=2$ and repeat step 4.
8. Let $k=i-1$, and check whether
$$4\gamma\bar{\mathcal{M}}_kEWE'\bar{\mathcal{M}}_k + (K_k-K_{k+1})'R (K_k-K_{k+1}) - 4\gamma\bar{\mathcal{M}}_{k+1}EWE'\bar{\mathcal{M}}_{k+1} \geq 0. \tag{13}$$
9. If above condition is satisfied then $\bar{\mathcal{M}} + \gamma\bar{\mathcal{V}}$ exist.
10. Repeat step 4 with $i=i+1$.

In a numerical simulation, we may have to choose the condition (13) to be greater than equal to small negative number such as -1×10^7. If one is running this algorithm in a loop for a different γ values, use the last determined K value as the initial K_1 for the next γ value.

5 Comparison with Freiling, Lee, and Jank's Algorithm

In [2], Freiling *et al.* has a similar condition as the condition (12) which is repeated here with our notations:

$$0 \leq \gamma(Z_k - Z_{k+1})BR^{-1}B'\bar{\mathcal{V}}_{k+1} + \gamma\bar{\mathcal{V}}_{k+1}BR^{-1}B' (Z_k - Z_{k+1}) + (Z_k - Z_{k+1})BR^{-1}B'(Z_k - Z_{k+1}) \tag{14}$$

where $Z_k = \bar{\mathcal{M}}_k + \gamma\bar{\mathcal{V}}_k$.

The difference between (12) and (14) are due to the differences in updating K. Freiling *et al.* use K_{k-1} to find $\mathcal{M}_k$ and use K_{k-1} and $\mathcal{M}_{k-1}$ to find $\mathcal{V}_k$. Then K_k is updated using $\mathcal{M}_k$ and $\mathcal{V}_k$. On the other hand, we use K_k to find $\mathcal{M}_k$ and $\mathcal{V}_k$. Then update K_{k+1} using these values. In other words, K_{k-1} is used to update $\mathcal{M}_k$ and $\mathcal{V}_k$ in [2], but we use K_k to update $\mathcal{M}_k$ and $\mathcal{V}_k$. Intuitively this is analogous to the differences in the current estimate where measurements $y(k)$ up to and including the k-th instant is used, and the predictor estimate where measurements up to $y(k-1)$ is used in filtering theory. Now, we show that for same updating scheme of K, these conditions are equivalent.

Corollary 5.1 *Assume that the same indexes are used,* i.e., *let $v=k+1$ in Freiling's equations. Then the condition (12) is equivalent to the condition (14).*

Proof: The left hand side of the inequality (14) can be rewritten as

$$\begin{aligned} & -\gamma(K_k-K_{k+1})'B'\bar{\mathcal{V}}_{k+1} - \gamma\bar{\mathcal{V}}_{k+1}B(K_k-K_{k+1}) \\ & +(K_k-K_{k+1})'R(K_k-K_{k+1}) \\ = \; & -\gamma[(A+BK_k)'\bar{\mathcal{V}}_{k+1} + \bar{\mathcal{V}}_{k+1}(A+BK_k)] \\ & +\gamma[(A+BK_k)'\bar{\mathcal{V}}_{k+1} + \bar{\mathcal{V}}_{k+1}(A+BK_k)] \\ & +(K_k-K_{k+1})'R(K_k-K_{k+1}) \end{aligned} \tag{15}$$

Note that the signs of the above equations appears to be reversed with respect to the Freiling *et al.*'s results because they have the expression for $-\Gamma_v$. Using Freiling *et al.*'s Equations (2.9) and (2.10) in [2]; $\Psi_v(K_{v-1}) = \gamma[(A+BK_{v-1})'\mathcal{V}_v + \mathcal{V}_v(A+BK_{v-1})] - Q$ and $\Psi_v(K_{v-1}) = -4\gamma\mathcal{M}_{v-1}W\mathcal{M}_{v-1} - Q$ with $v-1=k$ and $v=k+1$, we obtain, $-4\gamma\mathcal{M}_kEWE'\mathcal{M}_k = \gamma[(A+BK_k)'\mathcal{V}_{k+1} + \mathcal{V}_{k+1}(A+BK_k)]$ and $-4\gamma\mathcal{M}_{k+1}EWE'\mathcal{M}_{k+1} = \gamma[(A+BK_{k+1})'\mathcal{V}_{k+1} + \mathcal{V}_{k+1}(A+BK_{k+1})]$. Thus Equation (15) is equal to $4\gamma\mathcal{M}_kEWE'\mathcal{M}_k - 4\gamma\mathcal{M}_{k+1}EWE'\mathcal{M}_{k+1} + (K_k-K_{k+1})'R(K_k-K_{k+1})$ which gives the inequality (12). □

Even though updating $K_{k+1} = f(\bar{\mathcal{M}}_k + \gamma\bar{\mathcal{V}}_k)$ instead of $K_k = f(\mathcal{M}_k + \gamma\mathcal{V}_k)$ seems minor difference, the convergence results may differ between these two updating schemes, and an example of this will be shown in the next section.

6 Examples

Example 1: Consider a simple system given by $dx(t) = (-x(t) + k(t))dt + dw(t)$ with the cost $\hat{J}(t_F) = \int_0^{t_F}[x'(t)x(t) + k'(t)k(t)]dt$ and the covariance matrix $E\{dwdw'\} = 1dt$. For LQG case, we use the infinite time horizon version cost function, $J_{LQG} = \lim_{t_F\to\infty}(1/t_F)E\left\{\hat{J}(t_F)\right\}$. Solving for the positive real $\bar{\mathcal{P}}$ in the classical algebraice riccati equation, we obtain $\bar{\mathcal{P}} = -1\sqrt{2}$ and the controller $k(t) = (1-\sqrt{2})x(t)$. For MCV case, we use $J_{MCV} = \lim_{t_F\to\infty}(1/t_F)\left[E\left\{\hat{J}^2(t_F)\right\} - E^2\left\{\hat{J}(t_F)\right\}\right]$. Solving the coupled algebraic riccati equations (10) and (11), we obtain

$$0 = 12 - 8\sqrt{2+\gamma^2\bar{\mathcal{V}}^2} + (4\gamma^2 - 2\gamma)\bar{\mathcal{V}}^2 - 2(\sqrt{2+\gamma^2\bar{\mathcal{V}}^2})\bar{\mathcal{V}} \tag{16}$$

and

$$\bar{\mathcal{M}} = -1 + \sqrt{2+\gamma^2\bar{\mathcal{V}}^2}. \tag{17}$$

Thus, we choose γ and find all real positive $\bar{\mathcal{V}}$ using Equation (16), then we find corresponding $\mathcal{M}$ using Equation (17). For $\gamma \equiv 0$, we obtain the LQG results. When $\gamma = 0.1$ we obtain $\mathcal{M} = 0.41442$ and $\mathcal{V} = 0.23881$ from our algorithm and analytic solution. See Table 1. If we choose $\gamma = 1.6$, we have two possible solutions depending on the choice of $\mathcal{V}$.

Table 1: Comparison of the Algorithm solution with the Analytic Solution.

	Algorithm		Analytic	
γ	$\mathcal{M}$	$\mathcal{V}$	$\mathcal{M}$	$\mathcal{V}$
0.1	0.414	0.239	0.414	0.239
1.0	0.432	0.225	0.432	0.225
1.6	0.462	0.233	0.463 3.05	0.233 2.37

This example shows that there may be multiple positive definite solutions for a given γ. Thus, in general we cannot obtain uniqueness of the solutions for the coupled algebraic Riccati type equations. However, our algorithm can find one of the solutions.

Example 2: For the coupled algebraic Riccati equations (10) and (11), we repeat the example of Freiling *et al.* [2] using our algorithm. The constant matrices are given as

$$A = \begin{bmatrix} 1 & \frac{1}{8} \\ 0 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad E = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$Q = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The initial K_1 is chosen as

$$K_1 = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}.$$

We have tried the simulation with $\gamma = 3/8$ and $\gamma = 9/4$. For $\gamma = 3/8$ we obtained the results of Freiling *et al.* as expected, however, for $\gamma = 9/4$, Freiling conditions were not satisfied and the convergence was not guaranteed. But our condition (13) is satisfied and the convergence of $\mathcal{M} + \gamma\mathcal{V}$ is guaranteed. Unlike Freiling *et al.*'s theorem, we do not consider the monotonicity of $\bar{\mathcal{M}}$, we do however consider the monotonicity of $\mathcal{M} + \gamma\mathcal{V}$. We note that $\mathcal{M} + \gamma\mathcal{V}$ are monotonically decreasing as the Theorem 3.2 predicts because the condition (13) is satisfied.

For $\gamma = 3/8$ case, after 7 steps, the algorithm has reached the solutions

$$\begin{aligned} M_7 &= \begin{bmatrix} 3.2594 & 5.1900\times 10^{-2} \\ 5.1900\times 10^{-2} & 2.4175 \end{bmatrix} \\ V_7 &= \begin{bmatrix} 9.0695 & 1.6438\times 10^{-2} \\ 1.6438\times 10^{-2} & 2.1691 \end{bmatrix} \end{aligned}$$

For $\gamma = 9/4$ case, after 13 steps, the algorithm has reached the solutions

$$\begin{aligned} M_{13} &= \begin{bmatrix} 3.7542 & 6.4016\times 10^{-2} \\ 6.4016\times 10^{-2} & 2.4727 \end{bmatrix} \\ V_{13} &= \begin{bmatrix} 7.1655 & 1.1540\times 10^{-2} \\ 1.1540\times 10^{-2} & 1.8088 \end{bmatrix} \end{aligned}$$

Example 3: A roll/yaw attitude model of a geostationary satellite is represented as the following linear differential equation when $h_w >> \max\{I_i, \omega_c\}$,

$$\begin{aligned} dx(t) &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{h_w\omega_c}{I_1} & 0 & 0 & -\frac{h_w}{I_1} \\ 0 & -\frac{h_w\omega_c}{I_2} & \frac{h_w}{I_2} & 0 \end{bmatrix} x(t)\, dt \\ &\quad + \begin{bmatrix} 0 \\ 0 \\ \frac{B_{e0}}{I_1}\cos(\theta) \\ \frac{B_{e0}}{I_2}\sin(\theta) \end{bmatrix} m(t)\, dt + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{I_1} \\ \frac{1}{I_2} \end{bmatrix} dw(t) \end{aligned}$$

where h_w is the wheel momentum, ω_c is the orbital rate, I_i is the moment of inertia of the i-th axis, $x = [yaw, roll, yaw\_rate, roll\_rate]$ is the state, $m(t)$ is a dipole moment of the magnetic torquer (control), B_{eo} is the Earth's magnetic field strength, and $\frac{dw(t)}{dt}$ is the disturbance torque modeled as Gaussian white noise with intensity $0.7 \times B_{e0}$. The constants for the operational mode are given as $I_1 = 1988 kg\cdot m^2, I_2 = 1876 kg\cdot m^2, h_w = 55 kg\cdot m^2/s, \omega_c = 0.00418 deg/s, B_{eo} = 1.07\times 10^{-7} telsa$, and $\theta = 60 deg$ The initial condition is $[0.5deg,\ 0,\ 0,\ 0.007deg/s]$. For this illustration, we have chosen $Q = I$ and $R = 10^{-10}$.

Note that this example has two open loop poles at the imaginary axis, and these two pole locations cannot be changed using a feedback controller, K. Thus, this is a marginally stable case and we could not find K_1 that will stabilize $A + BK$. Thus, the Theorem 3.2 does not guarantee existence of a solution. However, if we turn off the existence condition (12), in 24 steps the algorithm finds the solutions for $\gamma = 250$.

$$M_{24} = \begin{bmatrix} -1.2856\times10^4 & -7.0469\times10^3 & -2.5509\times10^5 & 4.3927\times10^5 \\ -7.0469\times10^3 & -1.2813\times10^4 & -4.6276\times10^5 & 2.4099\times10^5 \\ -2.5509\times10^5 & -4.6276\times10^5 & -6.4926\times10^6 & 8.7114\times10^6 \\ 4.3927\times10^5 & 2.4099\times10^5 & 8.7114\times10^6 & -5.3298\times10^6 \end{bmatrix}$$

$$V_{24} = \begin{bmatrix} -2.1330\times10^1 & -1.9863\times10^1 & -7.2021\times10^2 & 7.2675\times10^2 \\ -1.9863\times10^1 & -4.2070\times10^1 & -1.5214\times10^3 & 6.7979\times10^2 \\ -7.2021\times10^2 & -1.5214\times10^3 & -3.6748\times10^4 & 2.4580\times10^4 \\ 7.2675\times10^2 & 6.7979\times10^2 & 2.4580\times10^4 & -7.5603\times10^3 \end{bmatrix}$$

7 Conclusions

The infinite time horizon MCV control solution is formulated for general nonlinear system with nonquadratic cost. Hamilton-Jacobi-Bellman equations are given for this general MCV formulation. For a linear system with quadratic cost function, coupled algebraic Riccati equations arise in the solution. The existence and uniqueness of these equations are considered. If we consider the coupled algebraic equations by themselve, we found a special condition which would guarantee existence of the solutions. For the MCV control problem, we found out that by keeping the mean cost at a prespecified level, we have existence and uniqueness of the solutions. The optimal performance happens when unique $\mathcal{M}^* + \gamma\mathcal{V}^*$ is determined. Furthermore the stabilizing controller is determined. Simple examples and a satellite attitude control application are provided to verify the existence theorem for the coupled algebraic Riccati equations.

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