Statistical Control for Performance Shaping using Cost Cumulants

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Abstract—The cost cumulant statistical control method optimizes the system performance by shaping the probability density function of the random cost function. For a stochastic system, a typical optimal control method minimizes the mean (first cumulant) of the cost function. However, there are other statistical properties of the cost function such as variance (second cumulant), skewness (third cumulant) and kurtosis (fourth cumulant), which affect system performance. In this paper, we extend the theory of traditional stochastic control by deriving the Hamilton-Jacobi-Bellman (HJB) equation as the necessary conditions for optimality. Furthermore, we derived the verification theorem for higher order statistical control, and construct the optimal controller. In addition, we utilize neural networks to numerically solve HJB partial differential equations. Finally, we provide simulation results for an oscillator system to demonstrate our method.

Index Terms—Cost cumulants, Hamilton-Jacobi-Bellman equation, neural networks, statistical control, stochastic optimization

I. INTRODUCTION

In statistical control paradigm, we consider the cost function as a random variable and use statistical controller to shape its probability distribution. Shaping of the cost distribution is thus achieved by optimizing the cost cumulants. In this way, we have the opportunity to influence and optimize the distribution of the cost function, which then in turn offers the possibility of controlling the probability of the occurrence of certain types of events. In intuitive terms, we refer to this as performance shaping. One instance of the statistical control is achieved by the selection of the mean, which is the first cumulant. In this case, the optimization of the mean of the cost, for a linear system with quadratic cost and Gaussian noise, leads to the well known linear-quadratic-Gaussian (LQG) control. Another instance of the choice of such a function is the selection of the cost variance, which is the second cumulant. This minimal cost variance (MCV) control is carried out while keeping the mean at a pre-specified level. Generalizing this idea, a finite linear combination of the cost cumulants is optimized in statistical control. The reader may wonder why we propose to use cost cumulants instead of cost moments. Even though moments and cumulants seem to be similar mathematically, in control engineering, cumulant control has shown substantial promise, especially in structural control [1], while cost moment control has been studied by various researchers without much promise. For example, even for a linear system, the optimization of first two moments lead to a nonlinear controller [2]. Moreover, low order cumulants are more significant than higher order cumulants, thus this leads to more effective numerical approximation algorithm [3].

Statistical control gives the controller the freedom to shape the cost distribution. The cost function is related to the performance of the system such as energy, fuel, or time. Consequently, by controlling the cost distribution, one can affect the performance of the system. For example, in satellite attitude control application, the pointing accuracy may be represented by the variance of the cost function. Consequently, instead of optimizing the mean pointing accuracy, one may minimize the variation using the second cumulant. Statistical control allows the control engineer to make those choices and shape the performance statistically through the cost cumulants.

Statistical control has been researched over the past five decades and it is closely related to Linear-Quadratic-Gaussian [4] and Risk Sensitive Control [5]. In typical Linear-Quadratic-Gaussian control, we find the controller such that the expected value of the utility function is optimized [6]. In Risk Sensitive control, denumerable sum of the quadratic cost functions is optimized [7], [5]. This idea can be generalized to the optimization of the arbitrary order cumulants of the cost function, which is the concept behind statistical optimal control [4], [8]. Consequently, statistical optimal control is a generalization of the traditional Linear-Quadratic-Gaussian (LQG) and risk-sensitive control.

Sain solved the open-loop minimum cost variance problem which is the second cumulant of a cost function [4]. In minimum cost variance control, the variance of the cost function is minimized, while the mean is constrained. Liberty et al. [9] studied minimum cost variance control for a linear system of quadratic cost function. Risk Sensitive control, as a special case of statistical control, was studied in [7], [10]. Sain et al. [11] analyzed the minimum cost variance control as an approximation of Risk Sensitive control where the mean and variance are optimized. In [3], we derived the HJB equations for 2nd and 3rd cumulant control problem. In [3] the method is based on iterative method and general n-th cost cumulant HJB equation, which is a necessary condition for optimality, is not given. By using an induction method, we derive n-th cost cumulant HJB equation in the current paper. Moreover, the optimal controller is not derived in [3]. Paper [12] is a preliminary conference version of this paper. There we did not have the induction proof for the HJB equation and the application was for a linear system. We find the 3rd cumulant control of a linear system in [12].

HJB equations are difficult to solve analytically, except for a few special cases such as Linear-Quadratic-Gaussian control. For a linear stochastic systems, previous works in [11], [13] solved the HJB equation for higher order cumulants. However, for a nonlinear system, there is no effective way to solve the corresponding HJB equations. Thus, we seek a method which handles both linear and nonlinear system HJB equations. Werbos proposed to use neural network for optimal control of a linear system in 1990 [14]. Beard proposed a method based on Galerkin approximation to solve the generalized HJB equation [15]. For discrete time nonlinear system, Chen and Jagannathan solved the HJB equation using neural networks [16]. For a deterministic continuous-time, nonlinear system, a neural network method is used to approximate the value function of the HJB equation in [17]. The neural network approximation method is similar to power series expansion method which was introduced by [18], [19]. This method approximates the value function in infinite-time horizon. The approximated value function and the optimal controller are determined by finding coefficients of the series expansion. In [20], the authors proposed a radial basis function neural network method for the output feedback system. Inspired by these advances, we develop a neural network method to solve HJB equations numerically for arbitrary order cost cumulant minimization problem. If we define the weighting and the basis functions as polynomials, the neural network is in fact equivalent to power series expansion in that we use the coefficients of the power expansion as the weights in neural networks. However, neural networks method has the potential to be more than simple power series expansion by adding more sophisticated layers of neurons and algorithms.

This paper is organized as follows. In Section II, the statistical...
control problem is formulated. In Section III, the necessary and sufficient conditions for the \( n \)-th cumulant optimization are addressed, and we construct the optimal controller in Section IV. In Section V, we introduce the notion of neural network method as a solution to the \( n \)-th cumulant HJB equations. In Section VI, simulation results are reported to show the effectiveness of our proposed neural network approach to the statistical control of a nonlinear system. Finally, we conclude the paper in Section VII.

II. Problem Formulation

The dynamic system that we are studying is given by the Ito-sense stochastic differential equation,

\[
\mathrm{d}x(t) = f(t, x(t), u(t, x(t)))\mathrm{d}t + \sigma(t, x(t))\mathrm{d}w(t),
\]

where \( t \in [t_0, t_F] \), \( x(t_0) = x_0 \), \( x(t) \in \mathbb{R}^n \) is the state, \( u(t) = k(t, x(t)) \in U \subset \mathbb{R}^m \) is defined as the system control, and \( \mathrm{d}w(t) \) is a Gaussian random process of dimension \( d \) with zero mean and covariance of \( W(t) \mathrm{d}t \). \( f \) and \( \sigma \) satisfy the local Lipschitz and linear growth conditions. A memoryless feedback control law is introduced as \( u(t) = k(t, x(t)) \) where \( k \) satisfies a Lipschitz condition. See p. 156 of [21] for details of this assumption. These conditions determine the nonlinear system that behaves like a linear system [22]. If such a control law \( k \) exists, then \( k \) is called an admissible control law. Next, we introduce a backward evolution operator \( \mathcal{O}(k) [11] \), defined by

\[
\mathcal{O}(k) = \frac{\partial}{\partial t} + \sum_{i=0}^{n} f_i(t, x, k(t, x)) \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{n} \left( \sigma(t, x)W(t)\sigma'(t, x) \right)_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}.
\]

Moreover, we use the Dynkin’s formula [23] in HJB equation derivations, as shown in the following equation.

\[
\Phi(t, x) = E \left\{ \int_{t_0}^{t_F} \mathcal{O}(k) \Phi(s, x) \mathrm{d}s + \Phi(t_F, x(t_F)) | x(t) = x \right\},
\]

where \( \Phi(t, x) \in C_{p,2}^{1,2}(\mathbb{Q}_0) \), \( \mathbb{Q}_0 = [t_0, t_F] \times \mathbb{R}^n \), \( \mathbb{Q}_0 \) is closure of \( \mathbb{Q}_0 \), \( k \) is admissible and \( E[|x(s)|^m | x(t) = x] \) is bounded for \( m = 1, 2, \ldots, n \) and \( t \in T \). In later derivations, we will use \( \varepsilon_{t_F} \) instead of \( E[|x(t) = x] \) to denote the conditional expectation.

The system cost

\[
J(t, x(t), k) = \int_{t_0}^{t_F} L(s, x(s), k(s, x(s))) \mathrm{d}s + \psi(x(t_F)),
\]

which is the quantity that we want to optimize, satisfies the conditions such that \( L(s, x(s), k(s, x(s))) \) is continuous, and \( L(s, x(s), k(s, x(s))) \) and \( \psi \) satisfy the linear growth condition. These conditions allow the mean of cost function, \( \varepsilon_{t_F}(J) \) to be finite. In order to mathematically define the \( n \)-th cumulant minimization problem, we first introduce the following definitions.

Definition 1 [11]: The \( i \)-th moment of the cost function is defined by the following Equation,

\[
M_i(t, x, k(t, x)) = E \left\{ J^i(t, x, k(t, x)) | x(t) = x \right\} = \varepsilon_{t_F}(J^i(t, x, k(t, x))), \quad i = 1, \ldots, n.
\]

Definition 2: A function \( M_i : \mathbb{Q}_0 \rightarrow \mathbb{R}^+ \) is an admissible \( i \)-th moment cost function if there exists an admissible control law \( k \) such that \( M_i(t, x, k(t, x)) = M_i(x,t) \).

Definition 3: Let the class of control law \( K_M \) be such that for each \( k \in K_M \), \( M_i \) satisfies Definition 2 for \( i = 1, 2, \ldots, n-1 \), where \( n \) denotes a fixed positive integer.

Definition 4: A function \( V_i : \mathbb{Q}_0 \rightarrow \mathbb{R}^+ \) is an admissible \( i \)-th cumulant cost function, if there exists an admissible control laws \( k \) such that \( V_i(t, x; k(t, x)) = V_i(t, x) \).

Definition 5 [24]: The admissible \( n \)-th cumulant of the cost function \( V_n \) is defined by the following Equation,

\[
V_n(t, x) = M_n(t, x) - \sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} M_{n-i-1}(t, x) V_{i+1}(t, x),
\]

where \( t \in [t_0, t_F] \), \( x(t_0) = x_0, x(t) \in \mathbb{R}^n \), \( M_i \) are admissible moment function for \( i = 1, 2, \ldots, n \), and \( V_i \) are admissible cumulant function for \( i = 1, 2, \ldots, n-1 \).

Consider an open set \( Q \subset \mathbb{Q}_0 \). Assume the cumulant cost function \( V_i(t, x) \in C_{p,2}^{1,2}(Q) \cap C(Q) \) are admissible functions for \( i = 1, 2, \ldots, n-1 \), where the set \( C_{p,2}^{1,2}(Q) \cap C(Q) \) means that the functions satisfy polynomial growth condition, and are continuous in the first and second derivatives of \( Q \), and continuous on the closure of \( Q \). This is restrictive assumption, which is only true for special cases. For non-smooth value function, viscosity method should be used as in [23]. This is reserved as the future study. We assume the existence of an optimal control law \( k \in K_M \), \( k \) be such that for each \( k \in K_M \), \( M_i \) satisfies Definition 2 for \( i = 1, 2, \ldots, n-1 \), and \( V_n \) is an admissible cumulant of the cost function \( V_n \) and satisfies the conditions such that

\[
V_n^*(t, x) = V_n(t, x; k_{n,M}^*(t, x)) = \min_{k \in K_M} \{ V_n(t, x; k(t, x)) \}, \quad (6)
\]

It should be noted that to find the optimal controller \( k_{n,M}^* \), we constrain the candidates of the optimal controller to \( K_M \), and the optimal \( V_n^*(t, x) \) is found with the assumption that lower order cumulants, \( V_i \), are admissible for \( i = 1, 2, \ldots, n-1 \).

III. \( n \)-th Cost Cumulant Minimization and HJB Equation

In this section, we mathematically analyze the \( n \)-th cost cumulant statistical control problem. The goal is to minimize the \( n \)-th cost cumulant of a cost function \( J \), subject to the system dynamics in the presence of the random noise. The following definition and lemmas are needed for deriving the \( n \)-th cost cumulant HJB equation.

Lemma 1: Let \( V_i(t, x), \ldots, V_{n-1}(t, x) \in C_{p,2}^{1,2}(Q_0) \cap C(Q_0) \) be admissible cost cumulant functions, then we have the following result (the arguments are suppressed)

\[
\frac{1}{2} \sum_{j=1}^{k-1} \frac{k!}{j!(k-j)!} \left( \frac{\partial V_j}{\partial x} \right) \sigma W \sigma' \left( \frac{\partial V_{k-j}}{\partial x} \right) = \sum_{i=0}^{k-2} \frac{(k-1)!}{i!(k-1-i)!} \left( \frac{\partial V_{i+1}}{\partial x} \right) \sigma W \sigma' \left( \frac{\partial V_{k-i-1}}{\partial x} \right).
\]

Proof: See Appendix A [25].

The necessary conditions for the \( n \)-th cost cumulant minimization problem are given through HJB equations. We use induction method to prove this theorem and it is a different approach than the iterative method of [26]. The following theorem is the main result of this paper.

Theorem 1: (Minimal \( n \)-th cumulant HJB equation)

Let \( V_1(t, x), V_2(t, x), \ldots, V_{n-1}(t, x) \in C_{p,2}^{1,2}(Q_0) \cap C(Q_0) \) be an admissible cumulant cost function. Assume the existence of an optimal control law \( k_{n,M}^* \in K_M \) and an optimal value function \( V_n^*(t, x) \in C_{p,2}^{1,2}(Q_0) \cap C(Q_0) \). Then the minimal \( n \)-th cumulant
cost function $V^*_n(x,t)$ satisfies the following HJB equation,
\[ 0 = \min_{k \in K(t,M)} \left\{ \mathcal{O}(k) [V^*_n(t,x)] + \frac{1}{2} \sum_{s=1}^{n-1} \frac{n!}{s!(n-s)!} \left( \frac{\partial V_n(t,x)}{\partial x} \right)^s \right\}, \]
for $(t,x) \in Q_0$, with the terminal condition $V^*_n(t_f,x) = 0$.

Proof: Here we present a brief proof of the theorem, because this is the main result of the paper. For a full version of the proof, see Appendix B. We use induction method to prove the theorem. We assume that the theorem holds for the second, third, ..., $(n-1)$-th cumulant case, then we will show that the theorem also holds for the $n$-th cumulant case.

Let $V^*_n$ be in the class of $C^{1,2}_{p_0}(Q) \cap C(\bar{Q})$. Apply the backward evolution operator $\mathcal{O}(k)$ to the recursive formula, we have
\[ \mathcal{O}(k) [V^*_n] = \mathcal{O}(k) [M_n] - \sum_{i=0}^{n-2} \frac{(n-1)!}{i!} \mathcal{O}(k) [M_{n-1-i} V_{n+i+1}] \]
(9)
Assume that $M_i(t,x) \in C^{1,2}_{p_0}(Q_0) \cap C(\bar{Q}_0)$ is the $j$-th admissible moment cost function. Then, $M_j(t,x)$ satisfy the following HJB equation,
\[ \mathcal{O}(k) [M_j(t,x)] + j M_{j-1}(t,x) L(t,x,k) = 0, \]
(10)
for $(t,x) \in Q_0$ where $M_{j-1}(t,x)$ is the $(j-1)$-th admissible moment cost function and the terminal condition is given as $M_j(t_f,x) = \psi(x(t_f))$ [3]. From (10), we have
\[ \mathcal{O}(k) [M_n] + n M_{n-1} = 0. \]
(11)
Let $M_i(t,x), V_i(t,x) \in C^{1,2}_{p_0}(Q) \cap C(Q)$, where $i$ and $j$ are non-negative integers, then
\[ \mathcal{O}(k) [M_i(t,x) V_j(t,x)] = \mathcal{O}(k) [M_i(t,x)] V_j(t,x) + M_i(t,x) \mathcal{O}(k) [V_j(t,x)] + \left( \frac{\partial M_i(t,x)}{\partial x} \right)^s \sigma(t,x) W \sigma(t,x)^s \left( \frac{\partial V_j(t,x)}{\partial x} \right), \]
(12)
with boundary condition $M_i(t_f,x) = \psi(x(t_f))$, $V_j(t_f,x) = \psi(t_f)$ [27].

By letting $i = n-1-j$, $j = i+1$ in (12), we obtain
\[ \mathcal{O}(k) [M_{n-1-j} V_{i+1}] = \mathcal{O}(k) [M_{n-1-j} V_{i+1}] + \left( \frac{\partial M_{n-1-j}}{\partial x} \right)^s \sigma W \sigma(t,x)^s \left( \frac{\partial V_{i+1}(t,x)}{\partial x} \right). \]
(13)
Substitute equations (11), (13) into (9), we have
\[ 0 = \mathcal{O}(k) [V^*_n] + n M_{n-1} - \sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ \mathcal{O}(k) [M_{n-1-i} V_{i+1}] \right. \]
\[ + \left. M_{n-1-i} \mathcal{O}(k) [V_{i+1}] + \left( \frac{\partial M_{n-1-i}}{\partial x} \right)^s \sigma W \sigma(t,x)^s \left( \frac{\partial V_{i+1}(t,x)}{\partial x} \right) \right]. \]
(14)
Using the fact that $M_0$ is an identity, and $\mathcal{O}(k) [V_1] = -L$, and the conversion formula between moment and cumulant [24]. Therefore, (14) becomes
\[ 0 = \mathcal{O}(k) [V^*_n] + \sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ M_{n-1-i} \mathcal{O}(k) [V_{i+1}] \right. \]
\[ + \left. \frac{(n-1)!}{i!(n-1-i)!} \left( \frac{\partial M_{n-1-i}}{\partial x} \right)^s \sigma W \sigma(t,x)^s \left( \frac{\partial V_{i+1}(t,x)}{\partial x} \right) \right]. \]
(15)
Then from Theorem 1, (7) and the formula in [28],
\[ \frac{\partial M_i}{\partial x} = \frac{i!}{j!(i-j)!} M_{i-j}, \]
equation (15) becomes
\[ 0 = \mathcal{O}(k) [V^*_n] - \sum_{i=1}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ M_{n-1-i} \sum_{j=0}^{i-1} \frac{j!}{j!(i-j)!} \right. \]
\[ \left. \left( \frac{\partial V_{i-j}}{\partial x} \right)^s \sigma W \sigma(t,x)^s \left( \frac{\partial V_{i-j+1}}{\partial x} \right) + \sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \right] \]
\[ \sum_{j=1}^{n-1-i} \frac{j!}{j!(n-1-i-j)!} \frac{1}{M_{n-1-i-j}} \left( \frac{\partial V_j}{\partial x} \right) \sigma W \sigma(t,x)^s \left( \frac{\partial V_{i-j+1}}{\partial x} \right). \]
(16)
From (16), we notice that both the second and third term on the right hand side contain the moment from $\{M_x\}$, where $x = 1, 2, \ldots, n-2$. Therefore, it is feasible to combine two summations with respect to $M_x$ and simplify the equation. Thus, (16) becomes
\[ 0 = \mathcal{O}(k) [V^*_n] - \sum_{i=1}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ M_{n-1-i} \sum_{j=0}^{i-1} \frac{j!}{j!(i-j)!} \right. \]
\[ \left. \left( \frac{\partial V_{i-j}}{\partial x} \right)^s \sigma W \sigma(t,x)^s \left( \frac{\partial V_{i-j+1}}{\partial x} \right) + \sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \right] \]
\[ \sum_{j=1}^{n-1-i} \frac{j!}{j!(n-1-i-j)!} \frac{1}{M_{n-1-i-j}} \left( \frac{\partial V_j}{\partial x} \right) \sigma W \sigma(t,x)^s \left( \frac{\partial V_{i-j+1}}{\partial x} \right) \right]. \]
(17)
In (17), for the second term, when $i$ varies from 1 to $n-2$, the corresponding $M_x$ varies from $M_{n-2}$ to $M_1$. In the third term, when $i$ varies from 0 to $n-2$, and $j$ varies from 1 to $n-2-i$, the corresponding $M_x$ also takes the value from $M_1$ to $M_{n-2}$. Now, we combine the $M_x$ terms from the second and third terms of (17). Therefore, (17) becomes the following equation,
\[ 0 = \mathcal{O}(k) [V^*_n] + \sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left( \frac{\partial V_{n-1-i}}{\partial x} \right)^s \sigma W \sigma(t,x)^s \left( \frac{\partial V_{i+1}}{\partial x} \right). \]
(18)
Using (7), (18) is written as
\[ 0 = \mathcal{O}(k) [V^*_n] + \frac{1}{2} \sum_{i=1}^{n-1} \frac{n!}{i!(n-1-i)!} \left( \frac{\partial V_{n-1-i}}{\partial x} \right)^s \sigma W \sigma(t,x)^s \left( \frac{\partial V_{i+1}}{\partial x} \right). \]

The theorem is proved.

Remarks. The HJB equation (8) in Theorem 1 provides a necessary condition for the optimality of the $n$-th cost cumulant statistical control. The optimality is achieved under the constraints that $V_1, V_2, \ldots, V_{n-1} \in C^{1,2}_{p_0}(Q) \cap C(\bar{Q})$ are admissible. Next, we will demonstrate the sufficient condition of the $n$-th cumulant optimality.

Theorem 2: (Minimal $n$-th cumulant HJB equation verification theorem)

Let $V_1(t,x), V_2(t,x), \ldots, V_{n-1}(t,x), M_1(t,x), M_2(t,x),$ $\ldots, M_{n-1}(t,x) \in C^{1,2}_{p_0}(Q) \cap C(\bar{Q})$ be an admissible cumulant function. Let $V^*_n \in C^{1,2}_{p_0}(Q) \cap C(\bar{Q})$ be
a solution to the partial differential equation,

\[
0 = \min_{k \in K_M} \left\{ \mathcal{O}(k) [V_n^{*}(t, x)] + \frac{1}{2} \sum_{i=1}^{n} \frac{n!}{i!(n-i)!} \left( \frac{\partial V_i}{\partial x} \right) \right\},
\]

(19)

with zero terminal condition. Then, \( V_n^{*}(t, x) \leq V_n(t, x; k) \) for all \( k \in K_M \) and \((t, x) \in Q\). If, in addition, there is a \( k \), that satisfies the following equation,

\[
\mathcal{O}(k) [V_n^{*}(t, x)] = \min_{k \in K_M} \left\{ \mathcal{O}(k) [V_n(t, x)] \right\}.
\]

(20)

Then \( V_n^{*}(t, x) = V_n(t, x; k) \), and \( k = k^* \) is an optimal controller.

Proof: See Appendix C. This theorem provides the sufficient condition of optimality for the \( n \)-th cumulant case. \( \square \)

Remarks: The control of the different cumulants leads to the different shapes of the distribution of the cost function. In Theorems 1 and 2, we generalize the traditional optimal control for the stochastic system, where the first cumulant (the mean value) of the cost function is minimized to the \( n \)-th cumulant control method, which minimizes any cumulant of a cost function. Therefore, we can control the distribution of the cost function and corresponding system performance by controlling specific order of the cumulant.

IV. OPTIMAL CONTROLLER FOR THE N-TH COST CUMULANT CONTROL

In this section, we seek a method to construct the \( n \)-th cumulant controller. From (1), we assume that

\[
f(t, x(t), k(t, x(t))) = g(t, x) + B(t, x)k(t, x),
\]

and from (3), we assume that

\[
L(s, x(s), k(s, x(s))) = l(s, x(s)) + k'(s, x(s))R(t)k(s, x(s)),
\]

where the matrices \( R(t) > 0 \), \( B(t, x) \) are continuous real matrices and \( l : Q_0 \rightarrow \mathbb{R}^+ \) is \( C^1(Q_0) \) and satisfies the polynomial growth conditions and \( g : Q_0 \rightarrow \mathbb{R}^n \) is \( C^1(Q_0) \) and satisfies linear growth condition. This special form for the control action allows us to find a linear controller even when the state is nonlinear with nonquadratic cost function.

Theorem 3: Assume the conditions in Theorem 2 are satisfied. The nonlinear optimal \( n \)-th cumulant controller is of the following form,

\[
k^*(t, x) = -\frac{1}{2} R^{-1} B' \left( \frac{\partial V_1}{\partial x} g_1 + \frac{\partial V_2}{\partial x} g_2 + \ldots + \frac{\partial V_n}{\partial x} g_n \right),
\]

(21)

with terminal condition \( V_n^{*}(t, x) = 0 \), which is the Lagrange multipliers.

Proof: See Appendix D. \( \square \)

Remarks: When we substitute \( k^* \) back to the HJB equation for the \( n \)-th cumulant and the partial differential equations from the first to \((n-1)\)-th cumulants, we obtain \( n \) partial differential equations. By solving these equations, we obtain the optimal statistical control problem and determine the corresponding Lagrange multipliers \( \gamma_2 \) to \( \gamma_n \). In the next section, we will study the method to solve the HJB equations for the \( n \)-th cumulant statistical control problems.

V. SOLUTION OF HJB EQUATION USING NEURAL NETWORKS

In Section III, we proved that the \( n \)-th cumulant minimization problem is equivalent to solving HJB equation (8) and in Section IV, we constructed the \( n \)-th optimal controller. However, partial differential equation (8) is difficult to solve directly, especially for the nonlinear systems. We utilize the neural network (NN) method in [12] to approximate the value functions in (8), and then find the solution of the HJB equations.

Our new neural network algorithm utilized power series basis functions, where the coefficients or weights of the network are tuned to solve the HJB equations. In [20], the authors utilized a radial basis functions in their network. NN method based on power series function has an important property of differentiability and with time-varying weights can approximate time-varying continuous functions [29]. The activation basis functions are polynomials and the NN weights associated with the basis functions are optimized by solving the system minimization problem over a compact region. For detailed NN solution see [12].

VI. NUMERICAL EXAMPLES

In this section, we demonstrate our \( n \)-th cost cumulant statistical control using an example. We show that our approach yields stabilizing feedback control for the nonlinear system. In the example, the input functions for the value functions are formulated as in [16];

\[
\delta_L(x) = \sum_{i=1}^{M} \left( \sum_{j=1}^{n} x_j \right)^{2i},
\]

(22)

where \( M \) is an even-order of the approximation, \( L \) is the number of neural network functions, \( n \) is the system dimension. All the terms from the expansion of the polynomial (22) are used to represent \( \delta_L(x) \).

Oscillator System Application

We will study the third cumulant, \( V_3 \), statistical control for a two-dimensional nonlinear oscillator system given in [30]. The general nonlinear dynamics equations are

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= -x_1(t) \left( \frac{\pi}{2} + \arctan(5x_1(t)) \right) - \frac{5x_1^2(t)}{2(1 + 25x_1^2(t))} + 4x_2(t) + 3u(t),
\end{align*}
\]

(23)

where \( x_1(t) \) and \( x_2(t) \) are set of state vectors and \( u(t) \) is the control. The system in (23) is a deterministic system, thus we introduce Gaussian noise into the system to add the process noise. The stochastic system is represented as

\[
\begin{bmatrix}
\frac{dx_1(t)}{dt} \\
\frac{dx_2(t)}{dt}
\end{bmatrix} =
\begin{bmatrix}
x_1(t) \left( \frac{\pi}{2} + \arctan(5x_1(t)) \right) - \frac{5x_1^2(t)}{2(1 + 25x_1^2(t))} + x_2(t) \\
0 - 4x_2(t)
\end{bmatrix}
\]

\[+ \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} u(t)dt + Edw(t), \]

(24)

where the state variables \( \tilde{x} \) are defined as: \( \tilde{x}(t) = [x_1(t), x_2(t)] \).

Here, we minimize the third cumulant, \( V_3 \), while constraining the second cumulant, \( V_2 \) and first cumulant, \( V_1 \). We assume that \( E \) in (24) is a \( 2 \times 1 \) constant vector of ones. \( dw(t) \) in (24) is a Brownian motion with mean \( E\{dw(t)\} = 0 \), and variance \( E\{dw(t)dw(t)\} = W = 0.1 \).
The cost function is quadratic and given as
\[
J(t, x(t), u(t)) = \int_{t_0}^{t_F} \left[ x_2^2(t) + u_1^2(t) \right] dt + \psi_1(x(t_F)),
\]
where \(\psi_1(x(t_F)) = 0\) is the terminal cost.

We will suppress the argument \(t\) for brevity. We utilize the neural network function defined in Section V to approximate the value functions in the HJB equation (8). We selected a polynomial function of sixth-order in the state variable (i.e., \(x\) is 6-th order) with length of the neural network series function, \(L = 15\) terms. These functions are given as follows.

\[
\begin{align*}
V_{1L}(x_1, x_2) &= w_{1}x_1^4 + w_{2}x_1^2 + w_{3}x_1^2 + w_{4}x_1^2 \\
&+ w_{5}x_1^2x_2^2 + w_{6}x_1x_2 + w_{7}x_1^2x_2 + w_{8}x_1^2x_2^2 \\
&+ w_{9}x_1^2 + w_{10}x_1^2 + w_{11}x_1^2x_2 + w_{12}x_1^2x_2 \\
&+ w_{13}x_1^2x_2^2 + w_{14}x_2x_1 + w_{15}x_2^2x_1 \\
V_{2L}(x_1, x_2) &= w_{16}x_1^4 + w_{17}x_1^2 + w_{18}x_1^2 \\
&+ w_{19}x_1^2 + w_{20}x_1x_2 + w_{21}x_1x_2 + w_{22}x_1x_2^2 \\
&+ w_{23}x_1^2x_2 + w_{24}x_2^6 + w_{25}x_2^6 + w_{26}x_2^6x_2 \\
&+ w_{27}x_1^2x_2 + w_{28}x_1^2x_2 + w_{29}x_2x_1 + w_{30}x_2x_1^2 \\
V_{3L}(x_1, x_2) &= w_{31}x_1^4 + w_{32}x_1^2 + w_{33}x_1^2 \\
&+ w_{34}x_1^2 + w_{35}x_1^2x_2 + w_{36}x_1x_2 + w_{37}x_1^2x_2 \\
&+ w_{38}x_1^2x_2^2 + w_{39}x_1^2 + w_{40}x_1^2 + w_{41}x_1x_2 \\
&+ w_{42}x_1^2x_2 + w_{43}x_1^2x_2 + w_{44}x_2x_1 + w_{45}x_2x_1^2
\end{align*}
\]

In the simulation, the asymptotic stability region for states was arbitrarily chosen as \(-4 \leq x_1 \leq 4\) and \(-3 \leq x_2 \leq 3\). The final time \(t_F\) is 30 s and we integrate backwards in time to solve for the weights. The initial state condition is given as \(x(0) = (3, -2)\).

From Fig. 1(a), we note that the neural network weights converge to constants, which are the optimal weights. Figs. 1(b) to 1(d) show the first, second and third cumulant value functions. From Fig. 1(b), it is observed that the value function \(V_1\) increases when the value of \(\gamma_2\) increases. For \(V_2\) and \(V_3\), from Figs. 1(c) and 1(d), the value functions show a decrease with increase in \(\gamma_2\). These graphs show that the control engineer can choose appropriate \(\gamma_2\) to shape the distribution through the \(1^{st}, 2^{nd}, 3^{rd}\) cumulants. Fig. 1(e) shows the state trajectory with noise influence with variance \(\sigma^2 = 0.1\). The states are bounded and show convergence to values close to zero. Fig. 1(f) shows the phase trajectory which indicates relatively low influence of the process noise, which has a small magnitude.

The optimal control trajectory, \(u\) is shown in Fig. 2 and converges to zero. It should be noted that the optimal controller is solved by selecting \(\gamma_2\) and \(\gamma_3\) where the value functions are minimum which in our case was \(\gamma_2 = 1\) and \(\gamma_3 = 0.1\) as shown in Figs. 1(b) to 1(d). In addition, we have the design freedom in \(\gamma\) values selection to enhance system performance.

VII. CONCLUSION

In this paper, we analyzed the statistical optimal control problem using a cost cumulant approach. The control of cost cumulants leads to the shaping of the distribution of the cost function, which improves the system performance. We developed the \(n\)-th cumulant control method which minimizes the cumulant of any order for a given stochastic system. The HJB equation for the \(n\)-th cumulant minimization is derived as necessary conditions of the optimality. The verification theorem, which is a sufficient condition, for the \(n\)-th cost cumulant case is also presented in this paper. We derived the optimal statistical controller for \(n\)-th cumulant statistical control. We used neural network approximation method to find the solutions of the HJB equation. Neural network method converts the HJB partial differential equation into the ordinary differential equation and solves them numerically. A nonlinear oscillator system is considered as an example. The results for the third cost cumulant statistical control is presented and discussed. From the results, we showed that a higher order cumulant statistical controller shapes the performance of the system through the cost cumulants.

REFERENCES

Appendix A
Proof of Lemma 1

Let $j = i + 1$, the right hand side of (7) becomes

$$
\sum_{j=1}^{k-1} \frac{(k-1)!}{(j-1)!}(k-j)! \left( \frac{\partial V_{k-j}}{\partial x} \right) \sigma W \sigma' \left( \frac{\partial V_j}{\partial x} \right).
$$

(29)

Then, when $k$ is odd, there are even number of terms in the above summation. We can rewrite the above summation in pairs. For example, the first term and last term, the second term and second last term, and so on. We represent the sum of each pair as follows:

$$
\frac{(k-1)!}{(j-1)!}(k-j)! \left( \frac{\partial V_{k-j}}{\partial x} \right) \sigma W \sigma' \left( \frac{\partial V_j}{\partial x} \right) + \frac{(k-1)!}{(j-1)!}(k-j)! \left( \frac{\partial V_j}{\partial x} \right) \sigma W \sigma' \left( \frac{\partial V_{k-j}}{\partial x} \right).
$$

Then, when $k$ is even, we have the same result except for the term when $j = \frac{k}{2}$.

In this case, we have

$$
\left( \frac{k-1}{k-2} \right) \left( \frac{k-1}{k-2} \right) \left( \frac{\partial V_{k/2}}{\partial x} \right) \sigma W \sigma' \left( \frac{\partial V_{k/2}}{\partial x} \right),
$$

which can be written as follows,

$$
\frac{(k-1)!}{(k-2)!} \left( \frac{\partial V_{k/2}}{\partial x} \right) \sigma W \sigma' \left( \frac{\partial V_{k/2}}{\partial x} \right).
$$

Therefore, the lemma (7) is proved [25].
Appendix B
Proof of Theorem 1

We use mathematical induction method assuming that the theorem holds for the second, third, . . . , (n − 1)-th cumulant case, then we will show that the theorem also holds for the n-th cumulant case.

Let V_n* be in the class of C^{1,2}_p(Q) \cap C(\hat{Q}). Apply the backward evolution operator \( O(k) \) to the recursive formula (5), we have

\[
O(k)[V_n^*] = O(k)[M_n] - \sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} O(k)[M_{n-1-i}V_{i+1}].
\]

(32)

Assume that \( M_j(t, x) \in C^{1,2}_p(Q_0) \cap C(\hat{Q}) \) is the j-th admissible moment cost function. Then, \( M_j(t, x) \) satisfy the following HJB equation,

\[
O(k)[M_j(t, x)] + J_{M_j-1}(t, x)L(t, x) = 0,
\]

(33)

for \( t, x \in Q_\theta \) where \( M_{j-1}(t, x) \) is the (j-1)-th admissible moment cost function and the terminal condition is given as \( M_j(t_F, x) = \psi(x(t_F)) \) [3]. From (33), we have

\[
O(k)[M_j] + nM_{j-1}L = 0.
\]

(34)

Let \( M_i(t, x), V_j(t, x) \in C^{1,2}_p(Q) \cap C(\hat{Q}), \) where i and j are non-negative integers, then

\[
O(k)[M_i(t, x)V_j(t, x)] = \left( \frac{\partial M_i(t, x)}{\partial x} \right)' \sigma(t) W_2(t, x)' \left( \frac{\partial V_j(t, x)}{\partial x} \right),
\]

(35)

with boundary condition \( M_i(t_F, x) = \psi(x(t_F)), V_j(t_F, x) = 0 \) [27]. By letting \( i = n-1-j, j = j+1 \) in (35), we obtain

\[
O(k)[M_{n-1-i}V_{i+1}] = O(k)[M_{n-1-i}]V_{i+1} + \left( \frac{\partial M_{n-1-i}}{\partial x} \right)' \sigma W \sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right).
\]

(36)

Substitute equations (34), (36) into (32), then we have

\[
0 = O(k)[V_n^*] + nM_{n-1}L + \sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} O(k)[M_{n-1-i}]V_{i+1} + M_{n-1-i}O(k)[V_{i+1}] + \left( \frac{\partial M_{n-1-i}}{\partial x} \right)' \sigma W \sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right).
\]

(37)

Use (34) again for \( O(k)[M_{n-1-i}] \) in (37) results in

\[
0 = O(k)[V_n^*] + nM_{n-1}L + \sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ O(k)[M_{n-1-i}]V_{i+1} - (n-1-i)M_{n-2-i}V_{i+1}L + M_{n-1-i}O(k)[V_{i+1}] \right] + \left( \frac{\partial M_{n-1-i}}{\partial x} \right)' \sigma W \sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right).
\]

(38)

From the third term of the right hand side of (38), we split the summation of the following term

\[
\sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ - (n-1-i)M_{n-2-i}V_{i+1}L \right],
\]

into two parts, one is the summation from \( i = 0 \) to \( i = n-3 \), the other is the term when \( i = n-2 \), such that

\[
\sum_{i=0}^{n-3} \frac{(n-1)!}{i!(n-1-i)!} \left[ - (n-1-i)M_{n-2-i}V_{i+1}L \right] = \sum_{i=0}^{n-3} \frac{(n-1)!}{i!(n-2-i)!} \left[ M_{n-2-i}V_{i+1}L \right] - (n-1)M_0V_{n-1}L.
\]

Furthermore, we split the summation

\[
\sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} M_{n-1-i}O(k)[V_{i+1}],
\]

into two parts such that

\[
\sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} M_{n-1-i}O(k)[V_{i+1}] = M_{n-1}O(k)[V_1] + \sum_{i=1}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ M_{n-1-i}O(k)[V_{i+1}] \right].
\]

Using the above results, the fact that \( M_0 \) is an identity, and \( O(k)[V_1] = -L \), we rewrite (38) as

\[
O(k)[V_n^*] + nM_{n-1}L - (n-1)V_{n-1}L - M_{n-1}L - \sum_{i=0}^{n-3} \frac{(n-1)!}{i!(n-2-i)!} \left[ M_{n-2-i}V_{i+1}L \right] + \sum_{i=1}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ M_{n-1-i}O(k)[V_{i+1}] \right] + \sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ \left( \frac{\partial M_{n-1-i}}{\partial x} \right)' \sigma W \sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right) \right] = 0.
\]

(39)

The second, third, fourth and fifth terms of (39) can be combined as

\[
(n-1)L \left[ M_{n-1} - \sum_{i=0}^{n-3} \frac{(n-1)!}{i!(n-2-i)!} \left[ M_{n-2-i}V_{i+1}L \right] - V_{n-1}L \right],
\]

which equals to zero according to the conversion formula between moment and cumulant [24]. Therefore, (39) becomes

\[
0 = O(k)[V_n^*] + \sum_{i=1}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ M_{n-1-i}O(k)[V_{i+1}] \right] + \sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ \left( \frac{\partial M_{n-1-i}}{\partial x} \right)' \sigma W \sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right) \right].
\]

(40)

For the second term in the right side of (40), we assume that Theorem 1 holds from the 2nd to (n-1)-th order cumulant case. We have

\[
\sum_{i=1}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ M_{n-1-i}O(k)[V_{i+1}] \right] = \sum_{i=1}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ \frac{i}{2} \sum_{j=1}^{i} \frac{i!}{j!(i+1-j)!} \right].
\]

(41)

Then, we use (7) to the above equation. The following equation is obtained,

\[
\sum_{i=1}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ M_{n-1-i}O(k)[V_{i+1}] \right] = \sum_{i=1}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ M_{n-1-i} \sum_{j=1}^{i} \frac{i!}{j!(i+1-j)!} \right].
\]

(42)

Using the formula in [28] that

\[
\frac{\partial M_i}{\partial V_j} = \frac{i!}{j!(i-j)!} M_{i-j}, \text{ then we have}
\]
\[
\frac{\partial M_k}{\partial x} = \sum_{j=1}^{i} \frac{\partial M_k}{\partial V_j} \frac{\partial V_j}{\partial x} = \sum_{j=1}^{i} \frac{i!}{j!(i-j)!} M_{i-j} \frac{\partial V_j}{\partial x}.
\]

Thus, (46) becomes
\[
O(k)[V_*] - \sum_{i=1}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ M_{n-i} \sum_{j=0}^{i-1} \frac{i!}{j!(i-j)!} \right] \left( \frac{\partial V_{i-j}}{\partial x} \right) W_{ij} \left( \frac{\partial V_{i-j+1}}{\partial x} \right) + \sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ \sum_{j=1}^{n-i-1} \frac{i!}{j!(i-j)!} M_{n-i-j} \left( \frac{\partial V_j}{\partial x} \right) W_{ij} \left( \frac{\partial V_{i-j+1}}{\partial x} \right) \right] + \sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ \left( \frac{\partial V_{i-j-1}}{\partial x} \right) W_{ij} \left( \frac{\partial V_{i-j+1}}{\partial x} \right) \right] = 0.
\]

In (48), for the second term, when \(i\) varies from 1 to \(n-2\), the corresponding \(M_k\) varies from \(M_{n-2}\) to \(M_1\). In the third term, when \(i\) varies from 0 to \(n-2\), and \(j\) varies from 1 to \(n-2-i\), the corresponding \(M_k\) also takes the value from \(M_1\) to \(M_{n-2}\). Now, we combine the \(M_k\) terms from the second and third terms of (48). From the second term of (48), we can expand the first summation in the following manner,
\[
\left. \left( \frac{\partial V_{i-j}}{\partial x} \right) W_{ij} \left( \frac{\partial V_{i-j+1}}{\partial x} \right) \right|_{i=1}^{n-2} = -\sum_{i=1}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ M_{n-i} \sum_{j=0}^{i-1} \frac{i!}{j!(i-j)!} \right] \left( \frac{\partial V_{i-j}}{\partial x} \right) W_{ij} \left( \frac{\partial V_{i-j+1}}{\partial x} \right) = -\sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ \sum_{j=1}^{n-i-1} \frac{i!}{j!(i-j)!} M_{n-i-j} \left( \frac{\partial V_j}{\partial x} \right) W_{ij} \left( \frac{\partial V_{i-j+1}}{\partial x} \right) \right] + \sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ \left( \frac{\partial V_{i-j-1}}{\partial x} \right) W_{ij} \left( \frac{\partial V_{i-j+1}}{\partial x} \right) \right] = 0.
\]

(49) Each term on the right hand side of (49) contains one of the moment term \(\{M_k\}\), where \(x = 1, 2, \ldots, n-2\). Then, we can find any specific \(M_k, k \in [1, n-2]\), by letting \(k = n-1-i\), which has the following form,
\[
\left( \frac{\partial V_{i-j}}{\partial x} \right) W_{ij} \left( \frac{\partial V_{i-j+1}}{\partial x} \right) = -\frac{(n-1)!}{i!(n-1-i)!} \sum_{j=0}^{n-i-1} \frac{i!}{j!(i-j)!} M_{n-i-j} \left( \frac{\partial V_j}{\partial x} \right) W_{ij} \left( \frac{\partial V_{i-j+1}}{\partial x} \right) + \sum_{j=0}^{n-i-1} \frac{i!}{j!(i-j)!} \left[ \left( \frac{\partial V_{i-j-1}}{\partial x} \right) W_{ij} \left( \frac{\partial V_{i-j+1}}{\partial x} \right) \right] M_k
\]

We examine the third term of (48) for the same \(M_k\), to see if we can combine the coefficients of \(M_k\) with the right hand side of (50). For the third term of (48), we rewrite this summation in terms of \(M_k\),
by letting \( k = n - 1 - i - j \), as follows,

\[
\sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-1-i)!} \left[ \sum_{j=1}^{n-2-i} \frac{(n-1-i)!}{j!(n-1-i-j)!} M_{n-1-i-j} \left( \frac{\partial V_i}{\partial x} \right) \sigma W\sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right) \right].
\]

The theorem is proved. \( \square \)

**APPENDIX C**

**PROOF OF THEOREM 2**

Assume there is a \( V_n^*(t, x) \in C_{p,2}^p(Q) \cap C(Q) \) which satisfies the following equation,

\[
0 = \min_{k \in \mathcal{K}_M} \left\{ \mathcal{O}(k) [V_n^*(t, x)] + \frac{1}{2} \sum_{i=0}^{n-1} \frac{n!}{i!(n-i)!} \left( \frac{\partial V_{i+1}}{\partial x} \right) \sigma(t, x) W\sigma(t, x)' \left( \frac{\partial V_{i+1}}{\partial x} \right) \right\}.
\]

Applying (7), to (54) is equivalent to

\[
0 = \min_{k \in \mathcal{K}_M} \left\{ \mathcal{O}(k) [V_n^*(t, x)] + \frac{1}{2} \sum_{i=0}^{n-1} \frac{n!}{i!(n-i)!} \left( \frac{\partial V_{i+1}}{\partial x} \right) \sigma(t, x) W\sigma(t, x)' \left( \frac{\partial V_{i+1}}{\partial x} \right) \right\}.
\]

Then, we substitute (9) into (55) and suppress the arguments, we have

\[
\min_{k \in \mathcal{K}_M} \left\{ \mathcal{O}(k) [M_n] - \sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-i-1)!} \mathcal{O}(k) [M_{n-1-i} V_{i+1}] \right\} = 0.
\]

Using the result of (11), (13), the second term in the curly bracket of the left hand side of (56) is written as

\[
\sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-i-1)!} \mathcal{O}(k) [M_{n-1-i} V_{i+1}]
\]

\[
= - \sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-i-1)!} [M_{n-2-i} V_{i+1} L] + \sum_{i=0}^{n-3} \frac{(n-1)!}{i!(n-i-1)!} \mathcal{O}(k) [V_{i+1}] + \sum_{i=0}^{n-2} \frac{(n-1)!}{i!(n-i-1)!} \left[ \left( \frac{\partial M_{n-1-i}}{\partial x} \right) \sigma W\sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right) \right].
\]

Then, we split the first term on the right hand side of (57) into two parts, one is from \( i = 0 \) to \( n-3 \), and the other is \( i = n-2 \). And we split the second term on the right hand side of (57) into two parts, one is \( i = 0 \), and the other is from \( i = 1 \) to \( n-2 \). We rewrite (57) as follows,

\[
\sum_{i=0}^{n-3} \frac{(n-1)!}{i!(n-i-1)!} \mathcal{O}(k) [M_{n-1-i} V_{i+1}]
\]

\[
= - \sum_{i=0}^{n-3} \frac{(n-1)!}{i!(n-i-1)!} [M_{n-2-i} V_{i+1} L] + \sum_{i=0}^{n-3} \frac{(n-1)!}{i!(n-i-1)!} \mathcal{O}(k) [V_{i+1}] + \sum_{i=0}^{n-3} \frac{(n-1)!}{i!(n-i-1)!} \left[ \left( \frac{\partial M_{n-1-i}}{\partial x} \right) \sigma W\sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right) \right].
\]
Furthermore, the term \(-(n - 1)V_{n-1}L - M_{n-1}L\) of equation (57) has the following property [24],
\[-(n - 1)V_{n-1}L - M_{n-1}L = \sum_{i=0}^{n-3} \frac{(n - 1)!}{i!(n - 2 - i)!} \]
\[M_{n-2-i}V_{i+1}L - nM_{n-1}L. \quad (59)\]

Therefore, we have
\[\begin{align*}
\sum_{i=0}^{n-2} \frac{(n - 1)!}{i!(n - 1 - i)!} & \mathcal{O}(k) [M_{n-1-i}V_{i+1}] \\
= - & \sum_{i=0}^{n-3} \frac{(n - 1)!}{i!(n - 2 - i)!} [M_{n-2-i}V_{i+1}L] \\
+ & \sum_{i=0}^{n-3} \frac{(n - 1)!}{i!(n - 2 - i)!} [M_{n-2-i}V_{i+1}L] - nM_{n-1}L \\
+ & \sum_{i=1}^{n-2} \frac{(n - 1)!}{i!(n - 1 - i)!} [M_{n-1-i}O(k) [V_{i+1}]] \\
+ & \sum_{i=1}^{n-2} \frac{(n - 1)!}{i!(n - 1 - i)!} \left( \frac{\partial M_{n-1-i}}{\partial x} \right) \sigma W \sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right) \\
= - & nM_{n-1}L + \sum_{i=0}^{n-2} \frac{(n - 1)!}{i!(n - 1 - i)!} [M_{n-1-i}O(k) [V_{i+1}]] \\
+ & \sum_{i=1}^{n-2} \frac{(n - 1)!}{i!(n - 1 - i)!} \left( \frac{\partial M_{n-1-i}}{\partial x} \right) \sigma W \sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right). \quad (60)
\end{align*}\]

Substituting (60) back to (56), we have
\[0 = \min_{k \in \mathcal{K}} \left\{ \mathcal{O}(k) [M_{n}] + nM_{n-1}L - \sum_{i=1}^{n-2} \frac{(n - 1)!}{i!(n - 1 - i)!} \right. \]
\[\left. [M_{n-1-i}O(k) [V_{i+1}]] \right. \\
- \sum_{i=0}^{n-2} \frac{(n - 1)!}{i!(n - 1 - i)!} \left( \frac{\partial M_{n-1-i}}{\partial x} \right) \sigma W \sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right) \right\} \quad (61)\]

From (42) and (43) in Theorem 1, we know the summation of the third and fourth term on the right hand side of (61) can be represented as the following,
\[\begin{align*}
\sum_{i=0}^{n-2} \frac{(n - 1)!}{i!(n - 1 - i)!} [M_{n-1-i}O(k) [V_{i+1}]] \\
+ & \sum_{i=0}^{n-2} \frac{(n - 1)!}{i!(n - 1 - i)!} \left[ \frac{\partial M_{n-1-i}}{\partial x} \right] \sigma W \sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right) \\
= - & \sum_{i=1}^{n-2} \frac{d}{d[l(i - j)!]} \left( \frac{\partial V_{n-i}}{\partial x} \right) \sigma W \sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right) \\
+ & \sum_{i=0}^{n-2} \frac{(n - 1)!}{i!(n - 1 - i)!} \left[ \sum_{j=1}^{n-1-i} \frac{(n - 1 - i)!}{j!(n - 1 - i - j)!} \right. \\
& \left. \left[ \frac{\partial V_{n-i}}{\partial x} \right] \sigma W \sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right) \right]. \quad (62)
\end{align*}\]

Furthermore, from (47), we notice that
\[\sum_{j=1}^{n-i} \frac{(n - 1 - i)!}{j!(n - 1 - i - j)!} M_{n-1-i-j} \left( \frac{\partial V_{n-i-j}}{\partial x} \right) \sigma W \sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right) \]
\[= \sum_{j=1}^{n-i} \frac{(n - 1 - i)!}{j!(n - 1 - i - j)!} M_{n-1-i-j} \left( \frac{\partial V_{n-i-j}}{\partial x} \right) \sigma W \sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right) \]
\[+ \left( \frac{\partial V_{n-i}}{\partial x} \right) \sigma W \sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right). \quad (63)\]

Substitute (63) into (62), we have
\[\begin{align*}
\sum_{i=1}^{n-2} \frac{(n - 1)!}{i!(n - 1 - i)!} [M_{n-1-i}O(k) [V_{i+1}]] \\
+ & \sum_{i=0}^{n-2} \frac{(n - 1)!}{i!(n - 1 - i)!} \left[ \left( \frac{\partial M_{n-1-i}}{\partial x} \right) \sigma W \sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right) \right] \\
= - & \sum_{i=0}^{n-2} \frac{(n - 1)!}{i!(n - 1 - i)!} \left[ M_{n-1-i} \sum_{j=1}^{i-1} \frac{1}{j!(i - j)!} \right. \\
& \left. \left( \frac{\partial V_{n-i-j}}{\partial x} \right) \sigma W \sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right) \right] + \sum_{i=0}^{n-2} \frac{(n - 1)!}{i!(n - 1 - i)!} \left[ \left( \frac{\partial M_{n-1-i}}{\partial x} \right) \sigma W \sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right) \right] \\
+ & \sum_{i=0}^{n-2} \frac{(n - 1)!}{i!(n - 1 - i)!} \left[ \left( \frac{\partial V_{n-i}}{\partial x} \right) \sigma W \sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right) \right]. \quad (64)
\end{align*}\]

From the proof of Theorem 1, we know that the sum of the second and third term of (48) is equal to zero. Furthermore, the first term on the right hand side of the above equation is same as the second term in (48); and the second term on the right hand side of the above equation is same as the third term in (48). Therefore, the summation of these two terms equals to zero, and (64) becomes,
\[\begin{align*}
\sum_{i=0}^{n-2} \frac{(n - 1)!}{i!(n - 1 - i)!} [M_{n-1-i}O(k) [V_{i+1}]] \\
+ & \sum_{i=0}^{n-2} \frac{(n - 1)!}{i!(n - 1 - i)!} \left[ \left( \frac{\partial M_{n-1-i}}{\partial x} \right) \sigma W \sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right) \right] \\
= & \sum_{i=0}^{n-2} \frac{(n - 1)!}{i!(n - 1 - i)!} \left[ \left( \frac{\partial V_{n-i}}{\partial x} \right) \sigma W \sigma' \left( \frac{\partial V_{i+1}}{\partial x} \right) \right]. \quad (65)
\end{align*}\]

Substitute (65) back to (61), then after manipulation, (61) becomes
\[\min_{k \in \mathcal{K}} \left\{ \mathcal{O}(k) [M_{n}(t, x)] + nM_{n-1}L \right\} = 0,
\]
which satisfies the sufficient condition for optimality for control in (33). Thus the theorem is proved. \(\square\)

APPENDIX D

PROOF OF THEOREM 3

From Theorem 1, the optimal controller \(k^*\) satisfies the following HJB equation,
\[0 = \min_{k \in \mathcal{K}} \left\{ \mathcal{O}(k) [M_{n}(t, x)] + \frac{1}{2} \sum_{s=1}^{n-1} \frac{n!}{s!(n - s)!} \right. \]
\[\left. \left( \frac{\partial V_{n-s}}{\partial x} \right) \sigma(t, x) W \sigma(t, x) \right. \]
\[\left. \left( \frac{\partial V_{n-s}}{\partial x} \right) \right\}, \quad (66)\]
with the constraint condition that the first, second, \ldots, \((n - 1)\)-th cumulant are admissible and satisfy the following partial differential
equations.

\[ O(k) [V_1(t,x)] + L(t,x,k) = 0, \]

\[ O(k) [V_n(t,x)] + \left( \frac{\partial V_n(t,x)}{\partial x} \right)' \sigma(t,x) W \sigma(t,x)' \left( \frac{\partial V_n(t,x)}{\partial x} \right) = 0, \]

\[ \vdots \]

\[ O(k) [V_{n-1}(t,x)] + \frac{1}{2} \sum_{s=1}^{n-2} \frac{(n-1)!}{s!(n-s-1)!} \left( \frac{\partial V_s(t,x)}{\partial x} \right)' \sigma(t,x) W \sigma(t,x)' \left( \frac{\partial V_{n-s-1}(t,x)}{\partial x} \right). \]

Therefore, we construct a new function \( F(t,x,k) \) such that

\[ F(t,x,k) = O(k) [V_n(t,x)] + \frac{1}{2} \sum_{s=1}^{n-1} \frac{(n)!}{s!(n-s)!} \]

\[ \left( \frac{\partial V_s(t,x)}{\partial x} \right)' \sigma(t,x) W \sigma(t,x)' \left( \frac{\partial V_s(t,x)}{\partial x} \right) \]

\[ + \lambda_1 \left( O(k) [V_1(t,x)] + L(t,x,k) \right) \]

\[ + \lambda_2 \left( O(k) [V_2(t,x)] + \left( \frac{\partial V_2(t,x)}{\partial x} \right)' \sigma(t,x) W \sigma(t,x)' \left( \frac{\partial V_2(t,x)}{\partial x} \right) \right) \]

\[ + \lambda_3 \left( O(k) [V_3(t,x)] + 3 \left( \frac{\partial V_3(t,x)}{\partial x} \right)' \sigma(t,x) W \sigma(t,x)' \left( \frac{\partial V_3(t,x)}{\partial x} \right) \right) \]

\[ \vdots \]

\[ + \lambda_{n-1} \left( O(k) [V_{n-1}(t,x)] + \frac{1}{2} \sum_{s=1}^{n-2} \frac{(n-1)!}{s!(n-s-1)!} \left( \frac{\partial V_s(t,x)}{\partial x} \right)' \sigma(t,x) W \sigma(t,x)' \left( \frac{\partial V_{n-s-1}(t,x)}{\partial x} \right) \right), \]

where \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{n-2} \) and \( \lambda_{n-1} \) are Lagrange multipliers.

Then, by using the method of Lagrange multiplier, we take the derivative of \( F(t,x,k) \) with respect to

\[ k, \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{n-2}, \lambda_{n-1}, \]

and equate result to zero. We obtain the following equations,

\[ \frac{\partial F(t,x,k)}{\partial k} = 2R \lambda_1 k + \lambda_1 B' \frac{\partial V_1(t,x)}{\partial x} + \lambda_2 B' \frac{\partial V_2(t,x)}{\partial x} + \lambda_3 B' \frac{\partial V_3(t,x)}{\partial x} + \ldots + \lambda_{n-1} B' \frac{\partial V_{n-1}(t,x)}{\partial x} + \lambda_n B' \frac{\partial V_n(t,x)}{\partial x} = 0, \]

\[ \frac{\partial F(t,x,k)}{\partial \lambda_1} = O(k) [V_1(t,x)] + L(t,x,k) = 0, \]

\[ \frac{\partial F(t,x,k)}{\partial \lambda_2} = O(k) [V_2(t,x)] \]

\[ + \left( \frac{\partial V_2(t,x)}{\partial x} \right)' \sigma(t,x) W \sigma(t,x)' \left( \frac{\partial V_2(t,x)}{\partial x} \right) = 0, \]

\[ \vdots \]

\[ \frac{\partial F(t,x,k)}{\partial \lambda_{n-1}} = O(k) [V_{n-1}(t,x)] + \frac{1}{2} \sum_{s=1}^{n-2} \frac{(n-1)!}{s!(n-s-1)!} \left( \frac{\partial V_s(t,x)}{\partial x} \right)' \sigma(t,x) W \sigma(t,x)' \left( \frac{\partial V_{n-s-1}(t,x)}{\partial x} \right) = 0. \]

We can calculate the optimal controller \( k^* \) from (69) as follows,

\[ k^* = - \frac{1}{2} R^{-1} B' \left( \frac{\partial V_1(t,x)}{\partial x} \right) \lambda_1 + \frac{\partial V_2(t,x)}{\partial x} \lambda_2 + \frac{\partial V_3(t,x)}{\partial x} \lambda_3 + \ldots + \frac{\partial V_{n-1}(t,x)}{\partial x} \lambda_{n-1} + \frac{\partial V_n(t,x)}{\partial x} \lambda_n. \]

Now, let

\[ \gamma_2 = \frac{\lambda_2}{\lambda_1}, \gamma_3 = \frac{\lambda_3}{\lambda_1}, \ldots, \gamma_{n-1} = \frac{\lambda_{n-1}}{\lambda_1}, \gamma_n = \frac{1}{\lambda_1}. \]

Then we have

\[ k^* = - \frac{1}{2} R^{-1} B' \left( \frac{\partial V_1(t,x)}{\partial x} + \gamma_2 \frac{\partial V_2(t,x)}{\partial x} + \gamma_3 \frac{\partial V_3(t,x)}{\partial x} + \ldots + \gamma_{n-1} \frac{\partial V_{n-1}(t,x)}{\partial x} + \gamma_n \frac{\partial V_n(t,x)}{\partial x} \right). \]

(70)