Nonlinear Orbital Dynamic Equations and State-Dependent Riccati Equation Control of Formation Flying Satellites[1]

Chang-Hee Won and Hyo-Sung Ahn

Department of Electrical Engineering
University of North Dakota

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Abstract

Precise maneuvers of formation flying satellites require a general orbital dynamic equation and an effective nonlinear control method. In this paper, nonlinear orbital dynamics of relative motion equations are derived for a constant distance separation formation flying problem. This general orbital dynamic equation allows elliptic, noncoplanar, and large separation distances between spacecrafts as well as traditional circular, coplanar, and small separation distance cases. Furthermore, for the in-plane formation flying scenario with large constant angle of separation between satellites, we derived the change in position and velocity equations. A nonlinear control method called the state-dependent Riccati equation control method is utilized to solve the formation flying control problem. This novel control method for a nonlinear system allows the intuitive design tradeoff between the control action and the state error similar to the

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classical linear-quadratic-regulator control method. Two numerical simulations demonstrate the effectiveness of the new state-dependent Riccati equation control method with the newly developed relative motion equations.

Introduction

Multiple spacecraft formation flying is one of the key technologies of current and future space missions. The National Aeronautical and Space Administration's Earth Observing-1 satellite demonstrated formation flying technology by flying in a formation one minute behind Landsat-7 in November 2000, and Jet Propulsion Laboratory is planning the Deep Space-3 mission to formation fly three separate spacecrafts to take space optimal interferometer measurements. The objective of formation flying is to autonomously control two or more satellites relative to another satellite with minimum ground control station involvement. There are two main challenges in formation flying technology. First, there has to be general nonlinear relative motion equation and second, there has to be an appropriate nonlinear control method to design an optimal controller. So, in this paper, we derive a nonlinear orbital dynamic equation and we propose to use the State-Dependent Riccati Equation (SDRE) control method to solve the formation flying problem.

On the orbital dynamic side of formation flying, various studies have been reported. In 1985, Vassar and Sherwood studied formation keeping for a pair of satellite in a circular orbit in spite of disturbances such as aerodynamic drag and solar radiation pressure [1]. Kapila *et al*. developed and used linear Clohessey-Wiltshire (CW) equation for circular orbit formation flying in their paper [2]. They utilized linear pulse control method in their example. In 2000, Yedavalli and Sparks studied satellite formation flying control design based on hybrid control system analysis [3]. They derived a nonlinear version of CW equation and linearized for small relation disturbance between two satellites. But they assumed circular orbits. Nonlinear CW equations for circular orbits have been extensively investigated by Afriend, Schaub, and Gim [4, 5]. Furthermore, Inalhan, *et al*. derived a nonlinear relative motion equation and solved the problem by linearizing with small curvature assumption [6]. The effect of nonlinearity and eccentricity are studied by Vaddi *et al.* [7]. Another popular model for formation flying orbital dynamics is leader following model of Wang and Hadaegh [8]. There, the leader satellite's frame becomes the formation reference frame. Then the states are measured and controlled with respect to that reference frame. Mesbahi and Hadaegh used leader following formulations, graphs, linear matrix inequalities, and switching theories [9]. However, the system is linear and with state-feedback control. Our relative motion equations are nonlinear equations that model elliptic, noncoplanar orbital dynamics of formation flying satellites.

Various control methods including bang-bang, linear pulsed control, linear quadratic regulator (LQR), and hybrid control have been studied for formation flying [1, 2, 3]. However, all these methods were based on a linear control theory with a linear system. In this paper, we utilize a nonlinear control method known as the SDRE control method.

There are many nonlinear control methods available in the literature including feedback linearization, gain scheduling, and sliding mode control, however, they determine the control law by some analysis method. Thus, it is difficult for a control designer to intuitively design a controller that is appropriate for the formation flying. One novel nonlinear control method called State-Dependent Riccati Equation (SDRE) control has the intuitive design advantages similar to the classical linear-quadratic-regulator (LQR) control method. The SDRE method is analogous to classical LQR method and the ability to tradeoff between control effort and state errors is also similar. However, this method does not guarantee global asymptotic stability and may give a suboptimal controller. The controllability and stability issues in SDRE control are addressed by Hammett *et al* [11]. In 1998 Mracek and Cloutier published a control design for the nonlinear benchmark problem using the SDRE method [12], where they showed the effectiveness of SDRE method. We will apply this novel SDRE method to formation flying satellite applications and demonstrate its effectiveness through simulations.

In the next section, we developed more general formation flying orbital dynamics for noncoplanar elliptic orbits that is valid for large separation distance between formation flying satellites. Then as a special coplanar case, we derive the equations necessary for constant angle separation formation flying in the same orbital plane. Then we describe the SDRE method in more detail including the stability properties. Finally, we present simulation results of multiple spacecraft flying in a formation using SDRE and newly developed relative motion equations.

Constant Separation Distance Orbital Dynamics

Here we consider noncoplanar and elliptic orbit geometry for constant separation distance formation flying. Consider an elliptic orbit with the Earth center at the one of the foci as shown in Fig. 1. An orthogonal coordinate frame, called the *s*-frame, is attached to the leader spacecraft and moves with the $\hat{i}$ -axis which is radially outward from the center of the earth. The $\hat{k}$-axis is out of the paper, and the $\hat{j}$-axis completes a right hand triad. The vector position of the following spacecraft is $\underline{r}^f = \underline{r}^l + \underline{\rho}$.

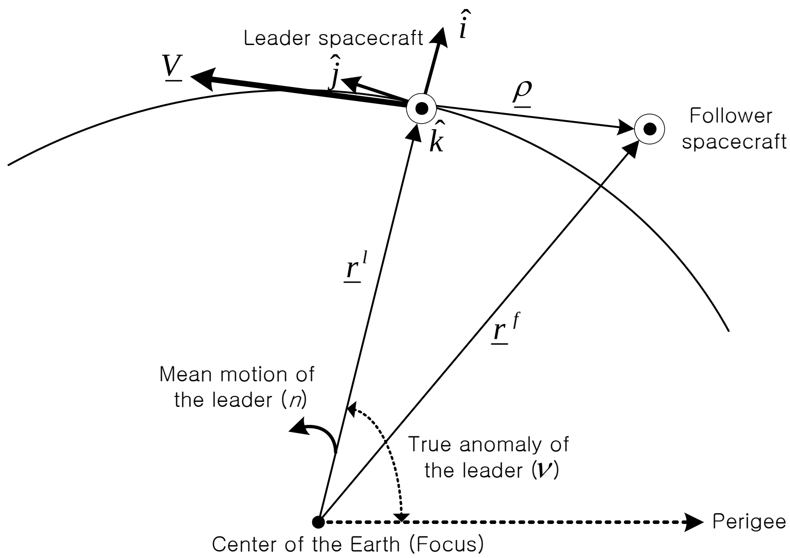

Fig. 1. Elliptic Orbital Geometry.

We will assume that the inertial reference frame, called *i*-frame, is attached to the Earth center, and the leader spacecraft is in our elliptic orbit around the Earth with the mean motion $n = \sqrt{\frac{\mu}{a^3}}$, where $\mu = 3.986 \cdot 10^{14}\, \mathrm{m^3/s^2}$ and a is the semi-major axis of the orbit. The orbital dynamics of a spacecraft relative to the Earth is given as

$$\ddot{\underline{r}}^l + \frac{\mu}{\left|\underline{r}^l\right|^3} \underline{r}^l = \underline{F}^l \ , \tag{1}$$

where $\underline{F}^l$ represents the specific external disturbance and/or specific control forces (forces per unit mass). If we let

$$\underline{\rho} = x\hat{i} + y\hat{j} + z\hat{k} \ , \tag{2}$$

then the follower spacecraft dynamic is given as

$$\ddot{\underline{r}}^l + \ddot{\underline{\rho}} + \frac{\mu}{\left|\underline{r}^l + \underline{\rho}\right|^3}(\underline{r}^l + \underline{\rho}) = \underline{F}^f \,, \tag{3}$$

where $\underline{F}^f$ denotes the specific external disturbances and/or specific control force (force per unit mass). For an elliptic orbit,

$$\underline{r}^l = \begin{bmatrix} r_x \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a(1-e^2)/(1+e\cos\nu) \\ 0 \\ 0 \end{bmatrix}. \tag{4}$$

The dynamic equation for the follower spacecraft with respect to the leader spacecraft can be written as

$$\ddot{\underline{\rho}} + \frac{\mu}{\left|\underline{r}^l + \underline{\rho}\right|^3}(\underline{r}^l + \underline{\rho}) - \frac{\mu}{r_x^3}\underline{r}^l = \underline{F}^f - \underline{F}^l \tag{5}$$

Define

$$\gamma = [(r_x + x)^2 + y^2 + z^2]^{3/2} \tag{6}$$

then $\left|\underline{r}^l + \underline{\rho}\right|^3 = \gamma$.

Now we need to transfer the acceleration $\underline{\rho}$ in the *s*-frame to the acceleration in the *i*-frame (inertial frame). The transformation is well known, for example see Wiesel's book [14].

$$^i\ddot{\underline{\rho}} = {}^s\ddot{\underline{\rho}} + 2\,{}^s\underline{\omega}^{si} \times {}^s\dot{\underline{\rho}} + {}^s\dot{\underline{\omega}}^{si} \times {}^s\underline{\rho} + {}^s\underline{\omega}^{si} \times [{}^s\underline{\omega}^{si} \times {}^s\underline{\rho}] \tag{7}$$

The second term on the right hand side is known as the Coriolis acceleration and the last term is known as centrifugal acceleration. $\underline{\omega}^{si}$ is the angular velocity of the leader satellite in the *s*-frame with respect to the *i*-frame,

$$\underline{\omega}^{si} = \begin{bmatrix} 0 & 0 & \dfrac{n(1+e\cos\nu)^2}{(1-e^2)^{3/2}} \end{bmatrix}^T \equiv \begin{bmatrix} 0 & 0 & \dot{\nu}_k \end{bmatrix}^T .$$

This angular velocity of the leader satellite varies with time. Consequently, we have

$$\underline{\dot{\omega}}^{si} = \begin{bmatrix} 0 & 0 & \dot{\nu}\dfrac{2n(1+e\cos\nu)(-e\sin\nu)}{(1-e^2)^{3/2}} \end{bmatrix}^T \equiv \begin{bmatrix} 0 & 0 & \ddot{\nu}_k \end{bmatrix}^T .$$

Also we have

$$^{s}\underline{\ddot{\rho}} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix}, \quad ^{s}\underline{\dot{\rho}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}, \quad ^{s}\underline{\rho} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{8}$$

Therefore we obtain

$$^{i}\underline{\ddot{\rho}} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} + \begin{bmatrix} -2\dot{\nu}\dot{y} \\ 2\dot{\nu}\dot{x} \\ 0 \end{bmatrix} + \begin{bmatrix} -\ddot{\nu}y \\ \ddot{\nu}x \\ 0 \end{bmatrix} + \begin{bmatrix} -\dot{\nu}^2 x \\ -\dot{\nu}^2 y \\ 0 \end{bmatrix} = \begin{bmatrix} \ddot{x} - 2\dot{\nu}\dot{y} - \ddot{\nu}y - \dot{\nu}^2 x \\ \ddot{y} + 2\dot{\nu}\dot{x} + \ddot{\nu}x - \dot{\nu}^2 y \\ \ddot{z} \end{bmatrix}. \tag{9}$$

Substitute Eq. (9) into Eq. (5),

$$\begin{bmatrix} \ddot{x} - 2\dot{\nu}\dot{y} - \ddot{\nu}y - \dot{\nu}^2 x \\ \ddot{y} + 2\dot{\nu}\dot{x} + \ddot{\nu}x - \dot{\nu}^2 y \\ \ddot{z} \end{bmatrix} + \frac{\mu}{\gamma}\left[\begin{pmatrix} r_x \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right] - \frac{\mu}{r_x^3}\begin{bmatrix} r_x \\ 0 \\ 0 \end{bmatrix} = \underline{F}^f - \underline{F}^l , \tag{10}$$

which simplifies to

$$\begin{bmatrix} \ddot{x} - 2\dot{\nu}\dot{y} - \ddot{\nu}y - \dot{\nu}^2 x + \dfrac{\mu}{\gamma}(r_x + x) - \dfrac{\mu}{r_x^2} \\ \ddot{y} + 2\dot{\nu}\dot{x} + \ddot{\nu}x - \dot{\nu}^2 y + \dfrac{\mu}{\gamma} y \\ \ddot{z} + \dfrac{\mu}{\gamma} z \end{bmatrix} = \begin{bmatrix} F_i \\ F_j \\ F_k \end{bmatrix}. \tag{11}$$

This is a general nonlinear equation of spacecraft relative position dynamics.

$F_i, F_j,$ and F_k are the resulting specific disturbance or control force. We can rewrite Eq. (11) in state-dependent coefficient form.

Let $x_1 = x,\ x_2 = \dot{x},\ x_3 = y,\ \dot{x}_4 = \dot{y},\ x_5 = z,\ x_6 = \dot{z}$, then we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \dot{\nu}^2 - \dfrac{\mu}{\gamma} & 0 & \ddot{\nu} & 2\dot{\nu} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\ddot{\nu} & -2\dot{\nu} & \dot{\nu}^2 - \dfrac{\mu}{\gamma} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\dfrac{\mu}{\gamma} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_i \\ F_j \\ F_k \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{\mu}{r_x^2} - \dfrac{\mu r_x}{\gamma} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

or

$$\underline{\dot{x}} \equiv A(x)\underline{x} + B(x)\underline{F} + E(x) . \tag{13}$$

This equation is a general nonlinear relative motion equation for formation flying satellites. Note that Eq. (12) is not a unique equation. It is important to note that this nonlinear relative motion equation did not assume that r_x is much greater than the relative distances $x,\ y,$ and z. Also this equation is valid for elliptic and noncoplanar orbits. However, to use SDRE method, we will assume $E(x) \approx 0$, and in the simulations we will check the validity of this assumption by explicitly calculating $E(x)$.

Furthermore, we can let x^f_{target} and $\dot{x}^f_{\text{target}}$ be the follower spacecraft's target position and speed in i-direction; y^f_{target} and $\dot{y}^f_{\text{target}}$ be the follower spacecraft's target position and speed in j-direction; and z^f_{target} and $\dot{z}^f_{\text{target}}$ be the follower spacecraft's target position and speed in k -direction. The state errors are defined by $\tilde{x}_1 = x_1 - x^f_{\text{target}}$, $\tilde{x}_2 = x_2 - \dot{x}^f_{\text{target}}$, $\tilde{x}_3 = x_3 - y^f_{\text{target}}$, $\tilde{x}_4 = x_4 - \dot{y}^f_{\text{target}}$, $\tilde{x}_5 = x_5 - z^f_{\text{target}}$, and $\tilde{x}_6 = x_6 - \dot{z}^f_{\text{target}}$. We will use the notation,

$$\underline{\tilde{x}} = \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 & \tilde{x}_4 & \tilde{x}_5 & \tilde{x}_6 \end{bmatrix}^T, \quad (14)$$

$$\underline{x}_{\text{target}} = \begin{bmatrix} x^f_{\text{target}} & \dot{x}^f_{\text{target}} & y^f_{\text{target}} & \dot{y}^f_{\text{target}} & z^f_{\text{target}} & \dot{z}^f_{\text{target}} \end{bmatrix}, \quad (15)$$

$$\frac{d}{dt}\underline{\tilde{x}} = A\underline{\tilde{x}} + A\underline{x}_{\text{target}} + B\underline{F}, \text{ and } \bar{u} = -K\underline{\tilde{x}}. \quad (16)$$

Coplanar Constant Angle Separation Orbital Dynamics

The objective of formation flying in this section is to achieve the constant angle between the leader and follower satellites in the same orbital plane. The relative position and relative velocity equations of the follower satellite with respect to the leader coordinate frame (*s*-frame) are derived. Fig. 2 shows the formation flying geometry with a constant separation angle, α, between the satellites. To keep this separation angle constant, the relative position and velocity are controlled.

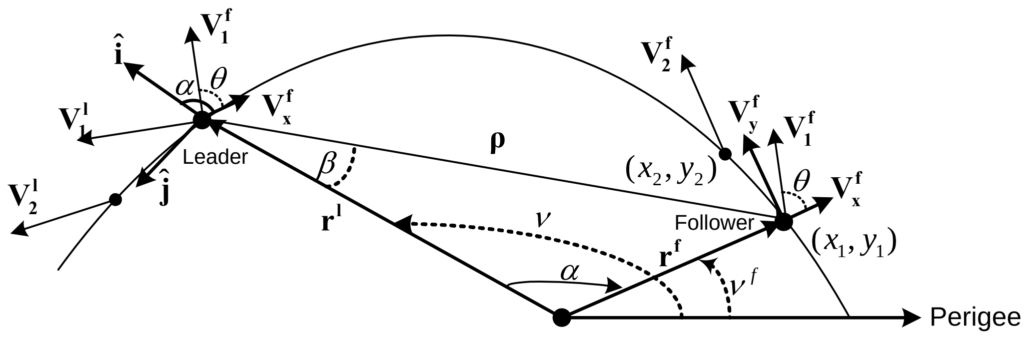

Fig. 2. Constant Angle Separation Formation Flying Geometry.

In Fig. 2, the distances from the center of the Earth to satellites, $\underline{r}^l$ and $\underline{r}^f$, are calculated from the true anomaly. To keep the angle, α, constant, we control the relative distance and velocities in the *s*-frame. The initial position coordinate, (x_1, y_1), of the follower satellite is the current position of the follower satellite with respect to the leader satellite *s*-frame. The target position coordinate, (x_2, y_2), of follower satellite is the target position of the follower satellite with respect to the moving leader satellite to keep the constant separation angle, α, in the same plane.

Now, we derive the relative position relationship. The relative position of the follower satellite with respect to the leader satellite in *s*-frame is given as follows.

$$\underline{x}(\nu) = -\rho(\nu)\cdot\cos(\beta(\nu))\,\hat{i}\text{, and }\underline{y}(\nu) = -\rho(\nu)\cdot\sin(\beta(\nu))\,\hat{j} \tag{17}$$

where, $\rho(\nu)$ denotes the magnitude of the vector $\underline{\rho}(\nu)$,

$$\rho(\nu) = \sqrt{r^{l2}(\nu) + r^{f2}(\nu) - 2r^l(\nu)r^f(\nu)\cos\alpha}\ ,\ \beta(\nu) = \sin^{-1}\left(r^f(\nu)\frac{\sin\alpha}{\rho(\upsilon)}\right),$$

$$r^l(\nu) = \frac{a(1-e^2)}{1+e\cos\nu}\text{ and } r^f(\nu) = \frac{a(1-e^2)}{1+e\cos(\alpha-\nu)}. \tag{18}$$

We derive the equation for the satellite's relative velocity in *s*-frame as

$$\Delta \underline{V}(\nu) = {}^{s}\underline{V}^{f}(\nu) - \underline{V}^{l}(\nu) . \qquad (19)$$

To find the follower satellite's velocity vector in the *s*-frame, ${}^{s}\underline{V}_{f}$, we transform the follower satellite's velocity vector to the *s*-frame.

$${}^{s}\underline{V}\ (\nu) = \begin{bmatrix} {}^{s}V_{x}^{f}(\nu) \\ {}^{s}V_{y}^{f}(\nu) \end{bmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{bmatrix} V_{x}^{\ f}(\nu) \\ V_{y}^{\ f}(\nu) \end{bmatrix} \qquad (20)$$

where $\begin{bmatrix} {}^{s}V_{x}^{f}(\nu) & {}^{s}V_{y}^{f}(\nu) \end{bmatrix}^{T}$ is the velocity vector of follower satellite with respect to the leader satellite's *s*-frame. Fig. 3 shows this relationship graphically. From Fig. 3, we know that $V_{x}^{f}(\nu) = V^{f}(\nu)\cos\theta$ and $V_{y}^{f}(\nu) = V^{f}(\nu)\sin\theta$ where $V^{f}(\nu)$ is the magnitude of the satellite's velocity vector to the s-frame, $V^{f}(\nu) = \left|{}^{s}\overline{V}^{f}(\nu)\right| = \sqrt{\left[V_{x}^{f}(\nu)\right]^{2} + \left[V_{y}^{f}(\nu)\right]^{2}}$. Then substituting these equations to Eq. (20), we obtain

$${}^{s}\underline{V}^{f}(\nu) = \begin{bmatrix} {}^{s}V_{x}^{f}(\nu) \\ {}^{s}V_{y}^{f}(\nu) \end{bmatrix} = \begin{bmatrix} V^{f}(\nu)\cos(\alpha-\theta) \\ -V^{f}(\nu)\sin(\alpha-\theta) \end{bmatrix}. \qquad (21)$$

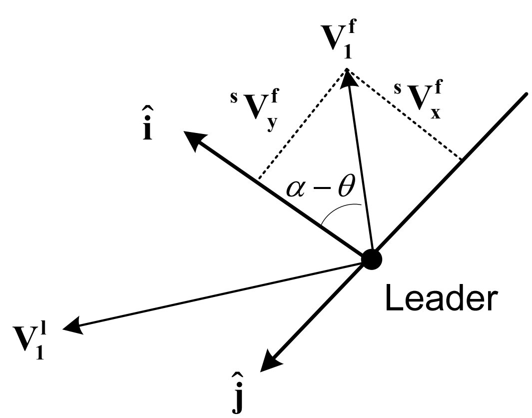

Fig. 3. Transformation of the Follower Satellite Velocity Vector to the *s*-Frame.

We can determine θ by $\theta = \tan^{-1}\left(\dfrac{V_y^f}{V_x^f}\right)$ where

$$V_x^f(\nu) = \sqrt{\frac{\mu}{a(1-e^2)}}(e\sin\nu^f),\text{ and } V_y^f(\nu) = \sqrt{\frac{\mu}{a(1-e^2)}}(1+e\cos\nu^f) \qquad (22)$$

where $\nu^f = \nu - \alpha$.

Next, we derive the equation for the leader satellite velocity vector, $\underline{V}^l$,

$$\underline{V}_x^l(\nu) = \sqrt{\frac{\mu}{a(1-e^2)}}\left(e\sin\nu\right)\hat{i},\text{ and } \underline{V}_y^l(\nu) = \sqrt{\frac{\mu}{a(1-e^2)}}(1+e\cos\nu)\,\hat{j}. \qquad (23)$$

So the relative velocity of follower satellite with respect to the leader satellite is

$$\Delta\underline{V}_x(\nu) = \left({}^sV_x^f(\nu) - V_x^l(\nu)\right)\hat{i},\text{ and } \Delta\underline{V}_y(\nu) = \left({}^sV_y^f(\nu) - V_y^l(\nu)\right)\hat{j}. \qquad (24)$$

These position and velocity vectors of Eq. (17) and Eq. (24) are controlled to keep a constant separation angle between the satellites.

If we assume that the true anomaly, ν, varies in discrete steps, then using Eqs. (17), (21), and (24) the target states are defined as

$$x_2(\nu_{k+1}) = -\rho(\nu_{k+1})\cdot\cos(\beta(\nu_{k+1})),\ y_2(\nu_{k+1}) = -\rho(\nu_{k+1})\cdot\sin(\beta(\nu_{k+1})),$$

$$\Delta V_{2x}(\nu_{k+1}) = V_2^f(\nu_{k+1})\cos(\alpha-\theta_{k+1}) - V_{2x}^l(\nu_{k+1}),$$

$$\Delta V_{2y}(\nu_{k+1}) = -V_2^f(\nu_{k+1})\sin(\alpha-\theta_{k+1}) - V_{2y}^l(\nu_{k+1}), \qquad (25)$$

and the initial states as

$$x_1(\nu_k) = -\rho(\nu_k)\cdot\cos(\beta(\nu_k)),\ y_1(\nu_k) = -\rho(\nu_k)\cdot\sin(\beta(\nu_k)),$$

$$\Delta V_{1x}(\nu_k) = V_1^f(\nu_k)\cos(\alpha-\theta_k) - V_{1x}^l(\nu_k),\text{ and } \Delta V_{1y}(\nu_k) = -V_1^f(\nu_k)\sin(\alpha-\theta_k) - V_{1y}^l(\nu_k).$$

(26)

The position and velocity change necessary to keep a constant separation angle are given as follows.

$$\Delta x = x_2(\nu) - x_1(\nu)\,,\ \Delta y = y_2(\nu) - y_1(\nu)\,,$$

$$\Delta V_x = \Delta V_{2x} - \Delta V_{1x}\,, \text{ and } \Delta V_y = \Delta V_{2y} - \Delta V_{1y}\,. \tag{27}$$

Here we are finding the difference between the next target position and velocity of the follower satellite and the current position and velocity of the follower satellite. We will try to find the optimal controller that minimizes this error. Finally, the next state errors in Eq. (14) are expressed as follows

$$\underline{\tilde{x}}(i+1) = \underline{\tilde{x}}(i) + \underline{\Delta}(i)\,, \tag{28}$$

where $\underline{\tilde{x}} = \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 & \tilde{x}_4 & \tilde{x}_5 & \tilde{x}_6 \end{bmatrix}^T$ and $\underline{\Delta} = \begin{bmatrix} \Delta x & \Delta V_x & \Delta y & \Delta V_y & \Delta z & \Delta V_z \end{bmatrix}$.

State-Dependent Riccati Equation Nonlinear Control Theory

The SDRE method started in 1996 by James Cloutier and his coworkers [10]. The SDRE control method reformulates a nonlinear dynamic equation into state-dependent coefficient structure and solves a SDRE as in classic linear quadratic regulator optimal control. It is intuitively similar to the LQR method in terms of the tradeoffs between the control effort and the errors. Also, the SDRE method has good robustness properties as the LQR method.

In this paper, to control the nonlinear system represented by the nonlinear orbital dynamic relative motion equation, we utilize the SDRE method. The SDRE concepts will be summarized in this section, and for more detailed description refer to Cloutier *et al* [10].

Consider a general nonlinear dynamic system.

$$\underline{\dot{x}} = f(\underline{x}) + g(\underline{x})\underline{u}\ , \text{ and } \ \underline{y} = C(\underline{x})\underline{x} \tag{29}$$

where $\underline{x} \in Ñ^n$, $\underline{\mathrm{u}} \in Ñ^m$. We assume that $f(0) = 0$ and $g(\underline{x}) \neq 0$ for all $\underline{x}$.

Remark: From Eq. (12) we note that for our problem $f(0) = 0$ is not generally satisfied, however, assuming $\underline{\rho}$ is small we can assume $f(0) = 0$.

The optimization problems is to find the control, $\underline{u}$, that minimize the cost function,

$$J = \frac{1}{2}\int_0^\infty \underline{x}^T Q(\underline{x})\underline{x} + \underline{u}^T R(\underline{x})\underline{u}\ dt \tag{30}$$

subject to the nonlinear differential equation, (29), where $Q(\underline{x}) \in C^k$, $\mathrm{R}(\underline{x}) \in C^k$, $Q(\underline{x}) \geq 0,\ \mathrm{R}(\underline{x}) > 0$ for all x. The SDRE method obtains a suboptimal solution of the above problem. First, put the nonlinear Eq. (29) to the state-dependent coefficient form,

$$\underline{\dot{x}} = A(\underline{x})\underline{x} + B(\underline{x})\underline{u}\ , \tag{31}$$

using direct parameterization, where $f(\underline{x}) = A(\underline{x})\underline{x}$ and $g(\underline{x}) = B(\underline{x})$. We note here that the choice of $A(\underline{x})$ is not unique, and this may lead to a suboptimal controller. Also this does not imply variations of the $A(x)$ matrix as a function of x, but we are simply putting $f(\underline{x})$ into a form that "looks" like a linear form.

Second, solve the SDRE state-dependent Riccati equation

$$Q(\underline{x}) + A^T(\underline{x})P(\underline{x}) + P(\underline{x})A(\underline{x}) - P(\underline{x})B(\underline{x})R^{-1}(\underline{x})B^T(\underline{x})P^T(\underline{x}) = 0 \tag{32}$$

to obtain $P(\underline{x}) \geq 0$. Third, obtain the nonlinear controller of the form.

$$\underline{u}(\underline{x}) = -R^{-1}(\underline{x})B^T(\underline{x})P(\underline{x})\underline{x} \tag{33}$$

We note that the Riccati matrix, $P(\underline{x})$ depends on the choice of $A(\underline{x})$, and because $A(\underline{x})$ is not unique, we have multiple suboptimal solutions [10, 13].

In addition Cloutier proved that if $A(\underline{x})$, $B(\underline{x})$, $Q(\underline{x})$, and $R(\underline{x})$ are smooth, and the pair $[A(x), B(x)]$ is pointwise stabilizable, and the pair $[C(x), A(x)]$ is detectable in the linear sense for all $\underline{x}$, then the SDRE method produces a closed loop solution which is locally asymptotically stable [10,12]. We can check stabilizability and detectability by forming the controllability and observability matrices and checking their rank.

Orbital Mechanics Preliminaries

Before we present the simulation results, we review some of the orbital mechanics concepts that are necessary to do the simulations.

A. Position of the Satellite as a Function of Time

We will find the position of the spacecraft as a function of time, $r_x(t)$. Assuming at $t = 0$ the spacecraft is at perigee, we determine eccentric anomaly, *E*, using the Kepler's equation.

$$M = E - e\sin E = nt \tag{34}$$

where *M* is the mean anomaly, n is the mean motion, t is time, and e is the eccentricity. In order to do this, we find the initial eccentric anomaly by

$$E_0 = \frac{M(1-\sin u) + u\sin M}{1+\sin M - \sin u}\ , \tag{35}$$

where $u = M + e$. We also have the following equations.

$$F(E_i) = E_i - e\sin E_i - M \text{ , and } \frac{dF(E_i)}{dE_i} = -e\cos E_i\,,$$

Then using Newton's method iteratively, we find E (final E_i is denoted by E) by

$E_{i+1} = E_i - \dfrac{F(E_i)}{dF/dE_i}$, and we determine

$$r_x(t) = a\frac{dF(E)}{dE} = a(1 - e\cos E)\,. \tag{36}$$

B. Orbit Perturbation Models

The oblateness (J2), solar radiation pressure, and air drag force are considered as the external disturbance. We will use the following simple models. These disturbances are added to the control force because $\underline{F}^l$ in Eq. (1) represents the specific control force and external disturbances.

Oblateness (J2)

The perturbing accelerations due to J_2 may be given in the ijk-coordinate system as [15]

$$f_{oi} = -\frac{3}{2}\frac{\mu J_2 R_E^{\,2}}{r_x^{\,4}}\left[1 - \frac{3}{2}\left(\sin^2 i\right)(1 - \cos 2u)\right] \tag{37}$$

$$f_{oj} = -\frac{3}{2}\frac{\mu J_2 R_E^{\,2}}{r_x^{\,4}}\left[\left(\sin^2 i\right)(\sin 2u)\right], \tag{38}$$

$$f_{ok} = -\frac{3}{2}\frac{\mu J_2 R_E^{\,2}}{r_x^{\,4}}\left[\sin i \cos i \sin u\right], \tag{39}$$

where argument of latitude is $u = \nu + \omega$ (true anomaly and argument of perigee), J_2 is equal to 0.00108263, i is orbital inclination, μ is gravitational constant, R_E is the equatorial radius, and r_x is the distance from the center of the Earth to the leader satellite.

Atmospheric Drag

The aerodynamic force per unit mass due to drag on a satellite is given by

$$f_d = 0.5\rho\ C_d \frac{S}{m} V^2 , \tag{40}$$

where ρ is atmospheric density, C_d is drag coefficient, S is satellite cross sectional area, m is mass, and V is satellite's velocity [16]. For the leader satellite in elliptic orbit, the velocity is given as $V_l = \dot{v} \cdot r_x$. And the follower satellite velocity is calculated by

$$V_f = \dot{v} \cdot r_x + \sqrt{x_2^2 + x_4^2 + x_6^2} \tag{41}$$

where r_x is given in Eq. (4) and x_2, x_4, and x_6 are given in Eq. (12). This drag force will affect the velocity in $-j$ direction only.

Solar Radiation Pressure

A simple model of solar radiation pressure (force per unit mass) is given as follows [14],

$$f_s = -4.5 \times 10^{-6} (1 + r) \frac{S}{m} , \tag{42}$$

where r is a reflection factor (the reflection factor is assumed 1 for the worst case), S is satellite cross sectional area exposed to the Sun in m^2, and m is satellite mass in kg. We will assume the model for summer solstice where the sun angle, θ, is 23.5 degrees. We will also assume that the satellite is at the perigee. The disturbance forces due to the solar radiation pressure on each axis of the s-frame are determined by using the information of orbit inclination angle and true anomaly angle as shown in Fig. 4. It is assumed that initially, the negative i -axis and the solar radiation incident direction after projected on the ij -plane are the same. Then as the satellite moves the i -axis moves with the true anomaly, ν , as shown in the top circle of Fig. 4.

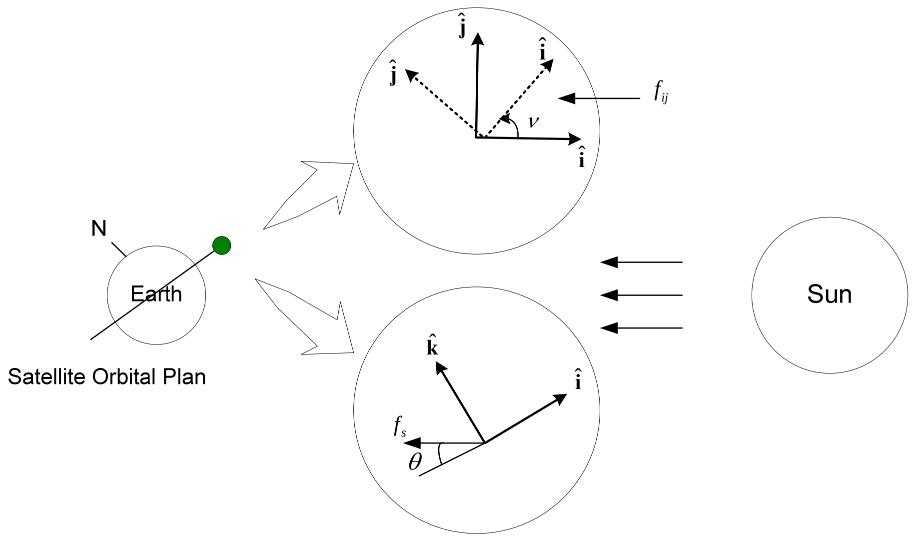

Fig. 4. Solar radiation Pressure Geometry.

From Fig. 4, we determine the solar radiation pressure in ij-frame as,

$$f_{ij} = f_s \cdot \cos(\theta) \tag{43}$$

and the solar radiation pressure in each of the s-frame is given as

$$f_i = -f_{ij} \cdot \cos(\nu),\ f_j = f_{ij} \cdot \sin(\nu),\ \text{and}\ f_k = f_s \cdot \sin(\theta) \tag{44}$$

where f_s is the solar radiation pressure from Eq. (37).

Formation Flying Simulations

In order to apply the newly developed formation flying orbital dynamic equations and

SDRE nonlinear control methods, we perform three different computer simulations. The first simulation is for satellites in Molynia orbit and -50 meters separation in all three ijk-directions. The second simulation is for a large separation distance formation flying with constant angle of separation equal to 120 degrees.

A. Molynia Orbit Formation Flying

In this simulation, two satellites are in Molynia orbit: semi-major axis of 32,978 km, eccentricity of 0.75, and inclination of 63.4 degrees. Thus this is formation flying in noncircular and noncoplanar orbit. Because we assume only 50-m separation distance, $\frac{\mu}{r_x^2} - \frac{\mu r_x}{\gamma}$ is very small, in the order of 10^{-6} to 10^{-7}, and the assumption $E(x) \approx 0$ is satisfied in Eq. (12). Furthermore, we check for the controllability of the system by check the rank of $[A(x), B(x)]$. Figure 8 shows the Molynia orbit with respect to the Earth.

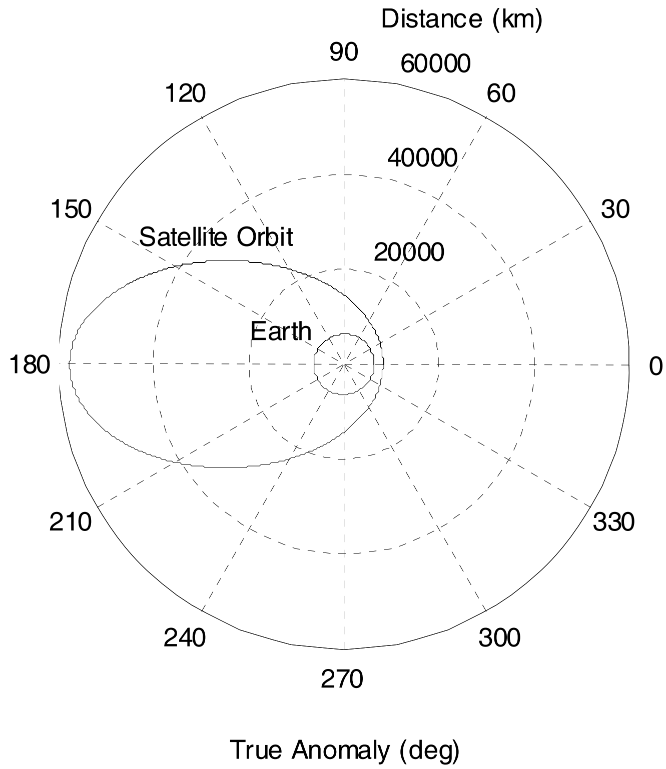

Fig. 8. Formation Flying Satellite Orbit with respect to the Earth.

Initially, both satellites will begin at the same position ($x = 0, y = 0,$ and $z = 0$), and then they will form the formation of 50 meters apart in all three directions of the *s*-frame (i.e, $x = -50, y = -50, z = -50$). We assumed summer solstice with the sun angle of 86.5 degrees. We used solar radiation pressure, air drag and J_2 perturbations. The simulation parameters are summarized in Table 1. The thrusters are assumed to be 1N thrusters in ijk directions. If the thrust force required to keep the formation is greater than 1N then only 1N is applied in the required direction.

Table 1. Satellite Parameters for Formation Flying in Molynia Orbit.

	Leader Satellite	Follower Satellite
Mass	1550 kg	410 kg
Dimension	$1.8 \times 1.8 \times 1.8$ m^3	$1.65 \times 1.65 \times 1.65$ m^3
Air drag coefficient	2.2	2.2
Solar reflectivity coefficient (r)	0.6	0.6

We chose the integration time step to be 0.5 second and simulation duration was 7 hours. We used the control weighting matrix, $R = diag\begin{bmatrix}1 & 1 & 1\end{bmatrix}$, and the state weighting matrix, $Q = diag\begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}$.

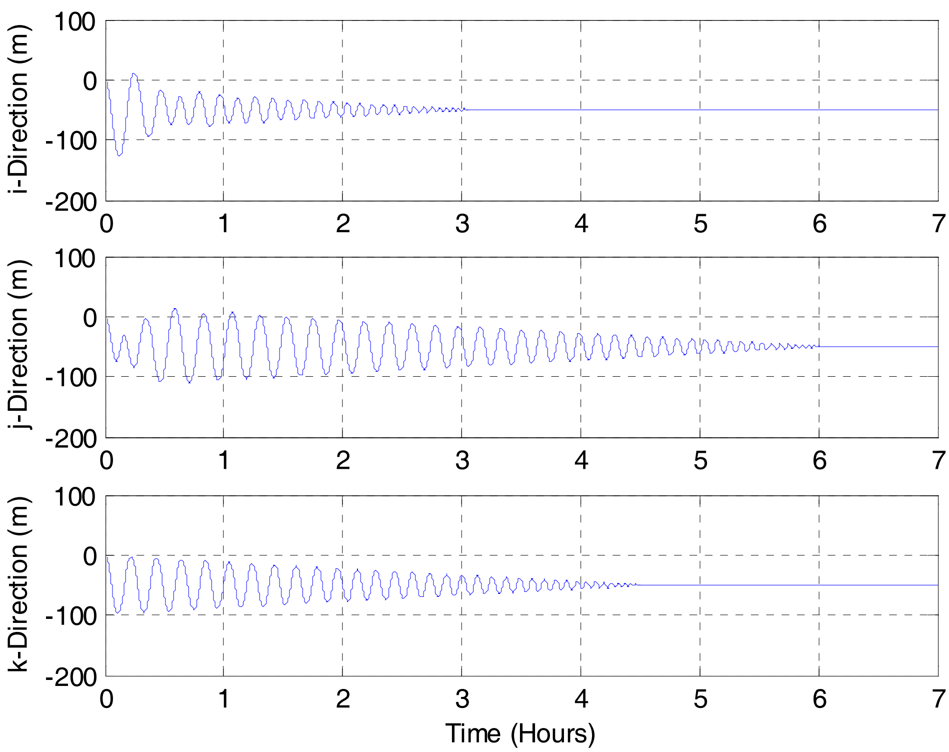

Fig. 9. Separation Distances of Two Formation Flying Satellites.

Figure 9 shows the separation distances decreases gradually to $-50\,\text{m}$ in ijk-directions. We note that in the i-direction the satellite reaches the target position of -50 meters first in about 3 hours, the k-direction second in about 4.5 hours, and the j-direction last in about 6 hours. The required time to reach -50 meters is large because we are limiting out control force to be 1N. Figure 10 shows the corresponding thrust pulses. We note that maximum thrust level of 1N was fired until the satellite reaches the desired formation.

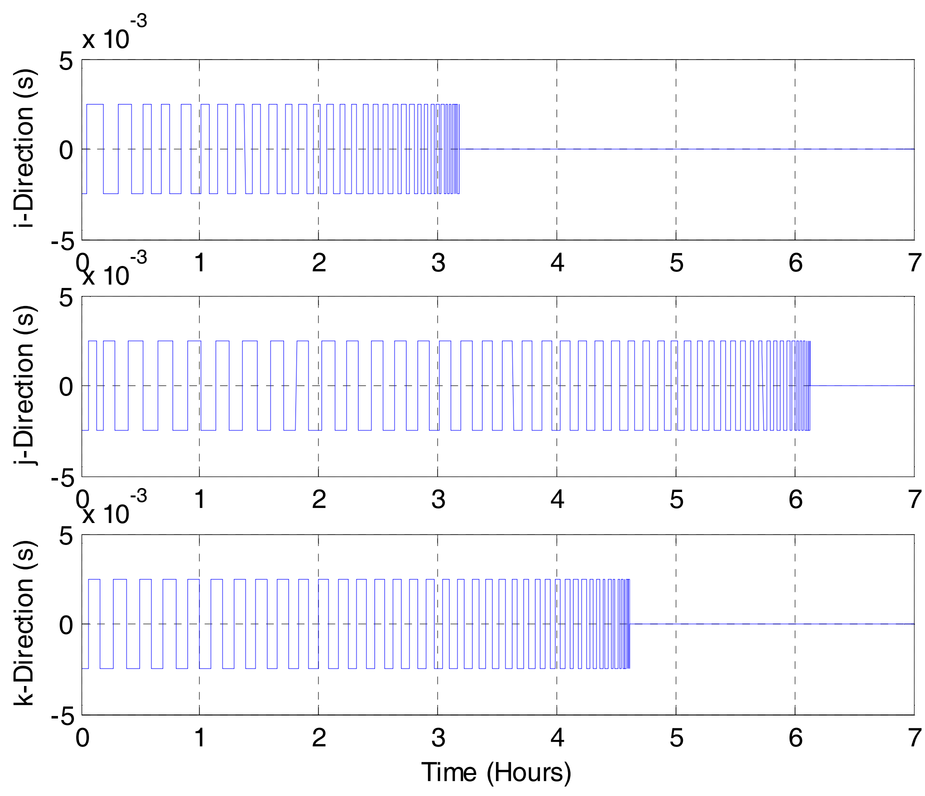

Fig. 10. Thrust Pulses of the Molynia Orbit Follower Satellite.

B. Constant Angle Separation Formation Flying in the Same Plane

In this simulation, two satellites are in low earth sun synchronous orbit: altitude of 800km, eccentricity of 0.04, and inclination of 98.6 degrees. These two satellites will keep a constant separation angle of 120 degrees with each other. We assumed summer solstice with the sun angle of 122.1 degrees. We used solar radiation pressure, air drag perturbations and J_2 perturbations. In this case the ideal thruster was assumed. We chose the integration time step to be 10 seconds and simulation duration was 1 hour. We used the control weighting matrix, $R = I_{3\times3}$, and the state weighting matrix, $Q = I_{6\times6}$.

The system is checked for the controllability in the simulation using Matlab "rank" and "ctrb" functions. In this simulation, the separation distance are large so $\frac{\mu}{r_x^2} - \frac{\mu r_x}{\gamma} \neq 0$ and the assumption $E(x) \approx 0$ is not satisfied in Eq. (12). The error is in the order of 1 to $1 \cdot 10^{-1}$. However, we will assume $E(x) \approx 0$ and use the SDRE method to control the formation. Thus we are finding the optimal controller for an approximate problem of $\underline{\dot{x}} \equiv A(x)\underline{x} + B(x)\underline{F}$.

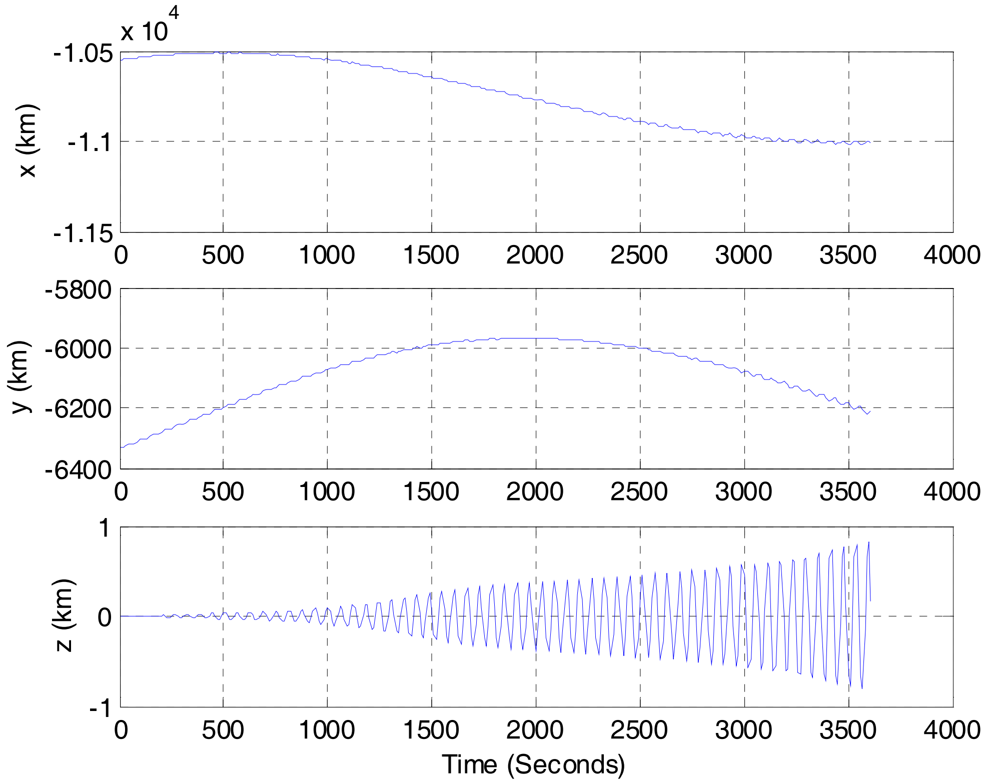

Fig. 11. Separation Distances of Constant Angle Formation Flying Satellites.

Figure 11 shows the relative distance between the leader and the follower satellites when the angle between them is kept at a constant. Note that the satellites move in a sinusoidal fashion in i-direction and j-direction. And the distance in k-direction is varying more rapidly. We are assuming 1N thrusters in all three directions. Because the thrust necessary to keep the formation is larger than 1N, the angle cannot be kept at 120 degrees for too long. The thrust necessary for this formation flying is shown in Figure 12. Note that we require a large amount of thrust to achieve this formation flying. Finally, Fig. 13 shows the separation angle between the two satellites. We are trying to keep this angle to 120 degrees in this formation flying simulation. Note that the separation angle is kept with ± 0.1 degrees for first 3600 seconds.

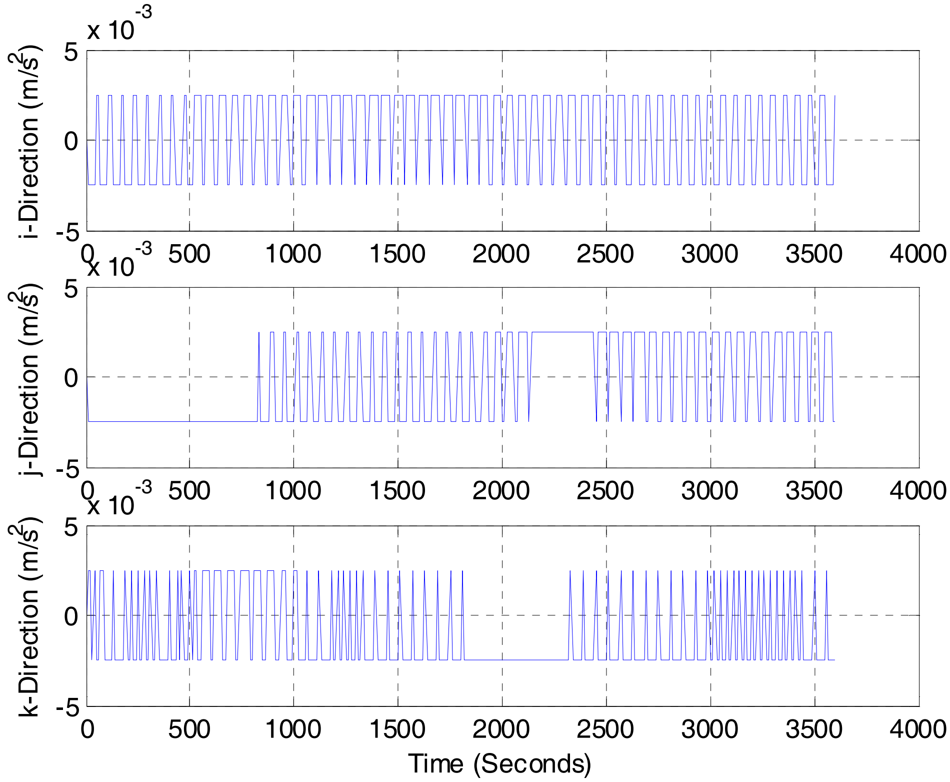

Fig. 12. Thrusts of the Constant Angle Formation Flying Follower Satellite.

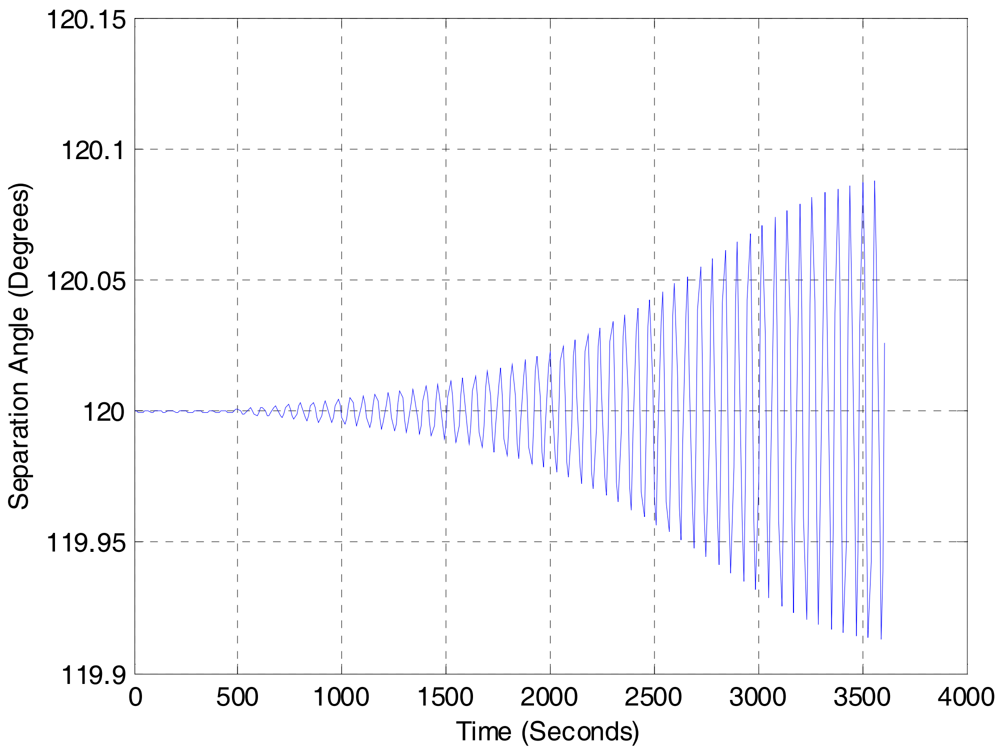

Fig. 13. Separation Angle Variation.

Conclusions

Nonlinear orbital dynamic equations for formation flying satellites are developed. These relative motion equations are valid for noncoplanar and elliptic formation flying. For the special case of in-plane formation flying with large angle of separations, we derived the change in position and velocity equations that can be used for formation flying. In order to control the satellites modeled by the newly developed nonlinear equations, we utilized the State-Dependent Riccati Equation (SDRE) control method. Computer simulations verified that the formation flying using the newly developed orbital dynamic equations and SDRE method gave favorable results.

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