I. Cost Cumulant and Risk-Sensitive Control

Cost Cumulant Control: Cost cumulant control is an optimal control method which minimizes a linear combination of quadratic cost cumulants.

Risk-Sensitive Control: Risk-sensitive control is an optimal control method which minimizes the exponential of the quadratic cost criterion. This is equivalent to optimizing an infinite linear combinations of all the cost cumulants.

Optimal control theory deals with the optimization, either minimization or maximization, of a given cost criterion. Linear-quadratic-Gaussian control, minimum cost variance control, and risk-sensitive control are discussed in terms of cost cumulants. Figure 1 presents an overview of the optimal control and the relationships among different optimal control methods.

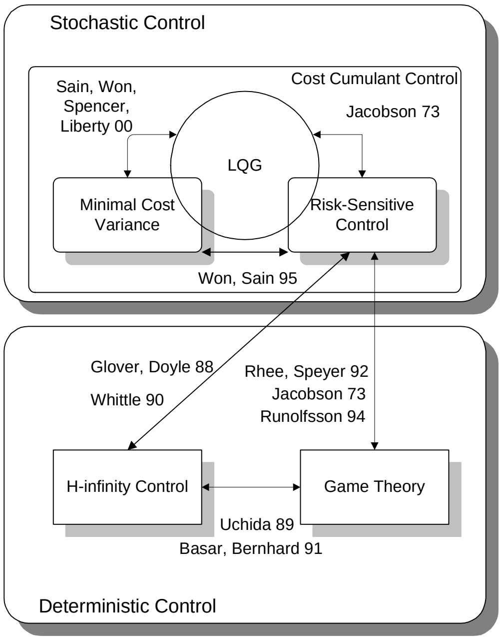

Fig. 1. Relationship between Various Optimal Control Methods

A. Linear-Quadratic-Gaussian Control

Linear quadratic Gaussian (LQG) control: LQG method optimizes the mean, which is the first cumulant, of a quadratic cost criterion [1][3][5].

Typical system dynamics for LQG control are given by the stochastic equation,

$$\begin{aligned} dx(t) &= Ax(t)dt + Bk(t,x)dt + E(t)dw(t) \\ y(t)dt &= Cx(t)dt + dv(t). \end{aligned} \tag{1}$$

Here $w(t)$ and $v(t)$ are vector Brownian motions. The meaning of a Brownian motion, such as $w(t)$, can be given directly or in terms of its differential $dw(t)$. In the latter case, we say that $dw(t)$ is a Gaussian random process with zero mean, covariance matrix Wdt, and independent increments. A similar description applies to $dv(t)$, with covariance Vdt. It is assumed that $dw(t)$ and $dv(t)$ are independent. The matrices A, B, C and E are of compatible size. It should be remarked that the formalism of (1) is that of a stochastic differential equation. Intuitively, one thinks of dividing both sides of this equation by dt, to obtain the more colloquial form. But the formal derivative of a Brownian motion–which is known as white noise–is not a well defined random process; and this motivates the alternate way of thinking.

The quadratic cost criterion is given by

$$J(k) = \int_0^{t_F} (x'Qx + k'Rk)\,dt. \tag{2}$$

The weighting matrices Q is symmetric and positive semidefinite matrix, and R is a symmetric and positive definite matrix. The LQG control problem then becomes a minimization of the mean quadratic cost over feedback controller k,

$$J^* = \min_k E\{J(k)\}. \tag{3}$$

The full-state-feedback control problem is to choose the control, k as a function of the state, x, so that the cost criterion (3) is minimized. The general partial observation or output feedback control problem is to choose the control, k as a function of the observation y so that the cost (3) is minimized.

Assume now that the problem has a solution of the quadratic form $\frac{1}{2}x'\Pi x$. The matrix Π can be found from the **Riccati equation**,

$$0 = \dot{\Pi}(t) + Q + A'\Pi(t) + \Pi(t)A - \Pi(t)BR^{-1}B'\Pi(t) \tag{4}$$

where $\Pi(t_F) = 0$.

Then the **full-state feedback optimal controller** is given by [3]

$$k(t, x) = -R^{-1}B'\Pi(t)x(t) \tag{5}$$

The solution of the output feedback LQG problem is found using the **certainty equivalence principle**. The optimal control is found using a Kalman filter where an optimal estimate $\hat{x}$ is obtained such that $E\{(x - \hat{x})'(x - \hat{x})\}$ is minimum. Then use this estimate as if it were an exact measurement of the state to solve the deterministic LQG control.

For the output feedback case, the estimated states are given by

$$\frac{d\hat{x}}{dt} = Ax + Bk + PCV^{-1}(y - C\hat{x}) \tag{6}$$

where P satisfies the forward Riccati equation,

$$\dot{P}(t) = W + AP(t) + P(t)A' - P(t)C'V^{-1}CP(t) \tag{7}$$

where the initial condition is $P(0) = \mathrm{cov}(x_0)$. Finally the optimal output feedback controller is given as

$$k(t,x) = -R^{-1}B'\Pi(t)\hat{x}(t). \tag{8}$$

B. Cost Cumulant Control

Minimal cost variance (MCV) control : MCV control is a special case of cost cumulant control where the second cumulant, variance, is minimized while the first cumulant, mean, is kept at a prespecified level.

Here open loop MCV and full-state-feedback MCV control laws are discussed. An open loop control law is a function $u : [0, t_F] \to \mathcal{U}$ where $\mathcal{U}$ is some specified allowable set of control values. A closed-loop or feedback control law is a function which depends on time and the past evolution of the process, i.e., $u(t, x(s); 0 \leq s \leq t)$.

B.1 Open Loop MCV Control

Consider a linear system [6]

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt + E(t)dw(t), \tag{9}$$

and the performance measure

$$J = \int_0^{t_F} [x'(t)Qx(t) + u'(t)Ru(t)]\, dt + x'(t_F)Q_F x(t_F) \tag{10}$$

where $w(t)$ is zero mean with white characteristics relative to the system, t_F is the fixed final time, $x(t) \in \Re^n$ is the state of the system, and $u(t) \in \Re^m$ is the control action. Note that

$$E\{dw(t)dw'(t)\} = W\ dt. \tag{11}$$

The fundamental idea behind minimal cost variance control is to minimize the variance of the cost criterion J

$$J_{MV} = VAR_k\{J\} \tag{12}$$

while satisfying a constraint

$$E_k\{J\} = M \tag{13}$$

where J is the cost criterion and the subscript k on E denotes the expectation based upon a control law k generating the control action $u(t)$ from the state $x(t)$ or from a measurement history arising from that state. By means of a Lagrange multiplier μ, corresponding to the constraint (13), one can form the function

$$J_{MV} = \mu(E_k\{J\} - M) + VAR_k\{J\}, \tag{14}$$

which is equivalent to minimizing

$$\hat{J}_{MV} = \mu E_k\{J\} + VAR_k\{J\}. \tag{15}$$

A Riccati solution to $\hat{J}_{MV}$ minimization is developed for the open loop case

$$u(t) = k(t,\ x(0)). \tag{16}$$

The solution is based upon the differential equations

$$\dot{z}(t) = A(t)z(t) - \frac{1}{2}B(t)R^{-1}B'(t)\hat{\rho}(t) \tag{17}$$

$$\dot{\hat{\rho}}(t) = -A'(t)\hat{\rho}(t) - 2Qz(t) - 8\mu Qv(t) \tag{18}$$

$$\dot{v}(t) = A(t)v(t) + E(t)WE'(t)y(t) \tag{19}$$

$$\dot{y}(t) = -A'(t)y(t) - Qz(t) \tag{20}$$

with boundary conditions

$$z(0) = x(0) \tag{21}$$

$$\hat{\rho}(t_F) = 2Q_F\ z(t_F) + 8\mu Q_F\ v(t_F) \tag{22}$$

$$v(0) = 0 \tag{23}$$

$$y(t_F) = Q_F\ z(t_F) \tag{24}$$

and the control action relationship

$$u(t) = -\frac{1}{2}R^{-1}B'(t)\hat{\rho}(t). \tag{25}$$

The variable $z(t)$ is the mathematical expectation of $x(t)$. The variable $\rho(t)$ corresponds to the costate variable of optimal control theory since it is the variable which enforces the differential equation constraint between $z(t)$ and $u(t)$. The variable $v(t)$ and $y(t)$ are introduced to reduce the integro-differential equation.

B.2 Full State Feedback Minimal Cost Variance Control

Consider the Ito sense stochastic differential equation (SDE) with control [7],

$$dx(t) = [A(t)x(t) + B(t)k(t,x)]\ dt + E(t)\ dw(t).$$

And the cost criterion

$$J(t,x(t),k) = \int_t^{t_F} [x(t)'Qx(t) + k'(t,x)R(t)k(t,x)]\, ds + x'(t_F)Q_F x(t_F). \tag{26}$$

In MCV control we define a class of admissible controllers, then the cost variance is minimized within that class of controllers. Define $V_1(t,x;k) = E\{J(t,x(t),k)|x(t) = x\}$

and $V_2(t,x;k) = E\{J^2(t,x(t),k)|x(t) = x\}$. A function M is an admissible mean cost criterion if there exists an admissible control law k such that

$$V_1(t,x;k) = M(t,x) \tag{27}$$

for all $t \in [0,t_F]$ and $x \in \Re^n$.

A minimal mean cost control law k_M^* satisfies $V_1(t,x;k_M^*) = V_1^*(t,x) \leq V_1(t,x;k)$, for $t \in T$, $x \in \Re^n$ and k an admissible control law. An MCV control law $k_{V|M}^*$ satisfies $V_2(t,x;k_{V|M}^*) = V_2^*(t,x) \leq V_2(t,x;k)$, for $t \in T$, $x \in \Re^n$ whenever k is admissible. The corresponding minimal cost variance is given by $V^*(t,x) = V_2^*(t,x) - M^2(t,x)$ for $t \in T$, $x \in \Re^n$. Here we present the full-state-feedback solution of the MCV control problem for a linear system and a quadratic cost criterion.

Then the linear optimal MCV controller is given by [7]

$$k_{V|M}^*(t,x) = -R^{-1}(t)B'(t)[\mathcal{M}(t) + \gamma(t)\mathcal{V}(t)]x,$$

where $\mathcal{M}$ and $\mathcal{V}$ are the solutions of the coupled Riccati-type equations (suppressing the time argument):

$$0 = \dot{\mathcal{M}} + A'\mathcal{M} + \mathcal{M}A + Q - \mathcal{M}BR^{-1}B'\mathcal{M} + \gamma^2\mathcal{M}BR^{-1}B'\mathcal{V} \tag{28}$$

and

$$0 \;=\; \dot{\mathcal{V}} + 4\mathcal{M}EWE'\mathcal{M} + A'\mathcal{V} + \mathcal{V}A - \mathcal{M}BR^{-1}B'\mathcal{V}$$

$$-\mathcal{V}BR^{-1}B'\mathcal{M}-2\gamma\mathcal{V}BR^{-1}B'\mathcal{V}, \tag{29}$$

with boundary conditions $\mathcal{M}(t_F)=Q_F$ and $\mathcal{V}(t_F)=0$. Once again, if γ approaches zero, we obtain classic LQG results.

C. Risk-Sensitive Control

A large class of control systems can be described in state variable form by the stochastic equations [2][8],

$$\begin{aligned} dx(t) &= Ax(t)dt+Bk(t,x)dt+dw(t) \\ y(t)dt &= Cx(t)dt+dv(t) \end{aligned} \tag{30}$$

Here $x(t)$ is a $2n-$dimensional state vector, $k(t,x)$ is an $m-$dimensional input vector, $w(t)$ is a $q-$dimensional disturbance vector of Brownian motions, $y(t)$ is a $p-$dimensional vector of output measurements, and $v(t)$ is an $r-$dimensional output noise vector of Brownian motions which affect the measurements being taken.

The risk-sensitive cost criterion is given by

$$J_{RS}(\theta)=-\theta^{-1}\log E_k\left\{e^{-\theta J}\right\} \tag{31}$$

where J is the classical quadratic cost criterion,

$$J = \int_{0}^{t_F} (x'Qx + k'Rk)\, dt. \tag{32}$$

The RS control problem then becomes a minimization of the cost $J_{RS}(\theta)$ over feedback controller k,

$$J^*_{RS}(\theta) = \min_u J_{RS}(\theta). \tag{33}$$

Assume a solution of the quadratic form $\frac{1}{2}x'\Pi x - \sigma' x + (\text{terms independent of } x)$. The matrix Π can be found from the Riccati-type equation,

$$0 = \dot{\Pi}(t) + Q + A'\Pi(t) + \Pi(t)A - \Pi(t)\left(BR^{-1}B' + \theta W\right)\Pi(t) \tag{34}$$

where $\Pi(t_F) = 0$.

Then the full-state feedback optimal controller is given by [8]

$$k(t, x) = -R^{-1}B'\Pi(t)x(t) + R^{-1}B'\sigma(t) \tag{35}$$

where $\dot{\sigma}(t) + (A - \Pi(B'R^{-1}B' + \theta W))'\ \sigma(t) = 0$ is a backward linear equation. The matrix P satisfies the forward Riccati-type equation,

$$\dot{P}(t) = W + AP(t) + P(t)A' - P(t)\left(C'V^{-1}C + \theta Q\right)P(t) \tag{36}$$

where $P(0) = \text{cov}(x_0)$. The updating equation for the risk-sensitive Kalman filter is given by

$$\frac{d\mathring{\mathrm{x}}}{dt} = Ax + Bk + PCV^{-1}(y - C\mathring{\mathrm{x}}) - \theta PQ\mathring{\mathrm{x}} \tag{37}$$

where $\mathring{\mathrm{x}}(0) = 0$. Finally the optimal output feedback controller is given as [8]

$$k(t, x) = -R^{-1}B'\Pi(t)\hat{x}(t) + R^{-1}B'\sigma(t) \tag{38}$$

where $\hat{x}$ is the minimal-stress estimate of x, given by

$$\hat{x}(t) = (I + \theta P(t)\Pi(t))^{-1}\left(\mathring{\mathrm{x}}(t) + \theta P(t)\sigma(t)\right). \tag{39}$$

As θ approaches zero, the cost criterion (31) become $E_k\{J\}$ and the matrices Π and P is obtained from the Riccati equation,

$$0 = \dot{\Pi}(t) + Q + A'\Pi(t) + \Pi(t)A - \Pi(t)BR^{-1}B'\Pi(t) \tag{40}$$

and

$$\dot{P}(t) = W + AP(t) + P(t)A' - P(t)C'V^{-1}CP(t). \tag{41}$$

Thus, we obtain the classic LQG result as θ approaches zero.

D. Relationship Between Risk-Sensitive and Cost Cumulant Control

To see the relationship between RS and cost cumulant control, consider a cost criterion,

$$J = \int_0^{t_F} \left[x(t)' Q x(t) + k'(t,x) R(t) k(t,x) \right] ds + x'(t_F) Q_F x(t_F). \tag{42}$$

Classical LQG control minimizes the first cumulant or the mean of the cost criterion (42). In MCV control we minimize the second cumulant of (42) while the mean is kept at a prespecified level. Furthermore, RS control minimizes an infinite linear combination of the cost cumulants. To see this, consider an RS cost criterion,

$$J_{RS} = -\theta^{-1} \log \left(E \left\{ \exp \left(-\theta J \right) \right\} \right), \tag{43}$$

where θ is a real parameter and E denotes expectation. Then the moment generating function or the first characteristic function is given by

$$\phi(s) = E \exp(-sJ). \tag{44}$$

The cumulant generating function $\psi(s)$ defined by

$$\psi(s) = \log \phi(s) = \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \beta_i s^i, \tag{45}$$

in which the $\{\beta_i\}$ are known as the **cumulants**, or sometimes the **semi-invariants**, of J. Now by comparing (43), (44), and (45), we note that

$$J_{RS} = \left(-\theta^{-1}\right)\left\{\sum_{i=1}^{\infty}\frac{(-1)^i}{i!}\beta_i(J)\,(\theta)^i\right\}, \tag{46}$$

where $\beta_i(J)$ denotes the i-th cumulant of J with respect to the control law k. Thus we note that the RS cost criterion is an infinite linear combination of the cost cumulants. Moreover, approximating to the second order,

$$\begin{aligned} J_{RS} &= \beta_1(J) - \frac{\theta}{2}\beta_2(J) + O(\theta^2) \\ &= E\{J\} - \frac{\theta}{2}VAR\{J\} + O(\theta^2). \end{aligned} \tag{47}$$

Therefore, we note minimal cost mean and minimal cost variance problems can be viewed as first and second order approximations of the RS control problem respectively. Minimizing the $VAR\{J\}$ under the restriction that the first cumulant $E\{J\}$ exists, is called the minimal cost variance (MCV) problem. Moreover minimizing any linear combination of cost cumulants under certain restrictions would be called cost cumulant control. Thus, classical LQG control (optimization of the first cumulant), MCV control (optimization of the second cumulant), and RS control (optimization of the infinite number of cumulants) are all special cases of the cost cumulant control problem.

E. Applications

An application of risk-sensitive control to satellite attitude maneuver is given in this part. An application of minimal cost variance control to an earthquake structure control is also given here. For linear quadratic Gaussian applications see [1][4][5]. For more risk-sensitive control examples refer to [2][8].

E.1 Risk-Sensitive Control Applied to Satellite Attitude Maneuver

This section shows the simulation results associated with the model of a geostationary satellite equipped with a bias momentum wheel on the third axis of body frame. This model assumes that the disturbance torque is Gaussian white noise. Then a stochastic RS controller is applied. For this model we assume small attitude angle, and roll/yaw dynamics are assumed to be decoupled from the pitch dynamics.

A roll/yaw attitude model of the geostationary satellite, is simplified as the following linear differential equation when, $h_w >> \max\{I_i, \omega_c\}$,

$$\begin{aligned} dx(t) &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{h_w \omega_c}{I_{11}} & 0 & 0 & -\frac{h_w}{I_{11}} \\ 0 & -\frac{h_w \omega_c}{I_{22}} & \frac{h_w}{I_{22}} & 0 \end{bmatrix} x(t)\, dt + \begin{bmatrix} 0 \\ 0 \\ \frac{B_e}{I_{11}} \cos(\alpha) \\ \frac{B_e}{I_{22}} \sin(\alpha) \end{bmatrix} m(t)\, dt \\ &\quad + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{I_{11}} \\ \frac{1}{I_{22}} \end{bmatrix} dw(t) & (48) \\ dy(t) &= I_{4\times 4} x(t)\, dt + dv(t) & (49) \end{aligned}$$

where $\frac{dw}{dt}$ is white Gaussian noise representing the disturbance torque, $\frac{dv}{dt}$ is white Gaussian noise representing the measurement noise, h_w is the wheel momentum, α is the angle that the positive roll axis makes with the magnetic torquer, ω_c is the orbital rate, I_{ii} is the moment of inertia of the i-th axis, $x = [\gamma, r, \dot{\gamma}, \dot{r}]$ is the state with yaw (γ) and roll (r), m is a dipole moment of the magnetic torquer (control), $B_e = 1.07 \times 10^{-7}$ telsa is the nominal magnetic field strength, and $I_{4\times4}$ is an identity matrix with dimension four. The expected value of $\frac{dw}{dt}$ is zero with $E\left\{\frac{dw}{dt}\frac{dw}{dt}'\right\} = 0.7B_e$ and the expected value of $\frac{dv}{dt}$ is zero with $E\left\{\frac{dv}{dt}\frac{dv}{dt}'\right\} = 1\times10^{-7}$. Here we chose $\theta = 5\times10^{-2}$ for the demonstration purpose, but this risk-sensitivity parameter, θ, should be viewed as another design parameter just like the weighting matrices Q and R. By varying this θ, we can obtain different performance and stability results. Theoretically, all θ that give a solution to the Riccati Equation (36) are possible. In the next example, we show how we choose this risk-sensitivity parameter to obtain larger stability margin. The constants for the operational mode are given as $I_{11} = 1988\ kg \cdot m^2, I_{22} = 1876\ kg \cdot m^2, I_{12} = I_{21} = 0, h_w = 55\ kg \cdot m^2/s, \omega_c = 0.00418$ deg/s, and $\theta = 60\ deg$. These values are actual parameters of the geostationary satellite. The initial condition is $[0.5\ deg,\ 0,\ 0,\ 0.007\ deg/s]$. Finally the weighting matrices are chosen to be $Q = I_{4\times4}$ and $R = 1 \times 10^{-10}$.

In this model, the states are measured with the sensor noise, $\frac{dv}{dt}$. Then a Kalman filter is used to estimate the states. The following simulations are performed using MATLAB, a software package. The RS controller is found using Equation (35). Note that both yaw and roll angles reduce to a value close to the origin. Figure 2 shows the roll and yaw angles with respect to time variation. After about 3 hours, both roll and yaw angles stay below 0.1 degree. Initially large control action is needed, but after 3 hours or so, less than 300 Atm^2 magnetic torque is required. It is important to note that despite the external

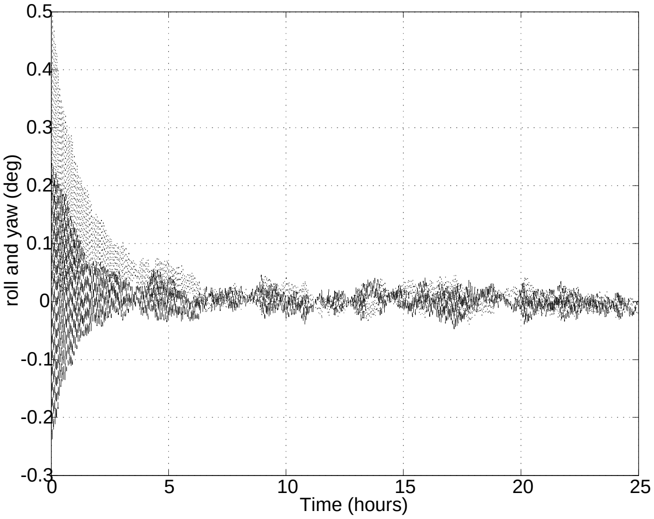

Fig. 2. Roll (dark) and Yaw (light) versus Time, RS Control

disturbances, RS control law produces good performance.

To compare the results with well known LQG controller, we simulated the system with an LQG controller. Here we let θ approach infinity in Equation (36). Note that Equation (36) becomes classical Riccati equation as θ goes to infinity. This is shown in Figure 3. Note that in LQG case, it takes longer for yaw and roll angles to fall below 0.1 degree, and the variation in the angles are larger than the RS case. Thus in this sense, RS controller outperforms the LQG controller.

E.2 MCV Control Applied to Seismic Protection of Structures

A 3DOF, single-bay structure with an active tendon controller as shown in Figure 4 is considered here. The structure is subject to a one-dimensional earthquake excitation. If we assume a simple shear frame model for the structure, then we can write the governing

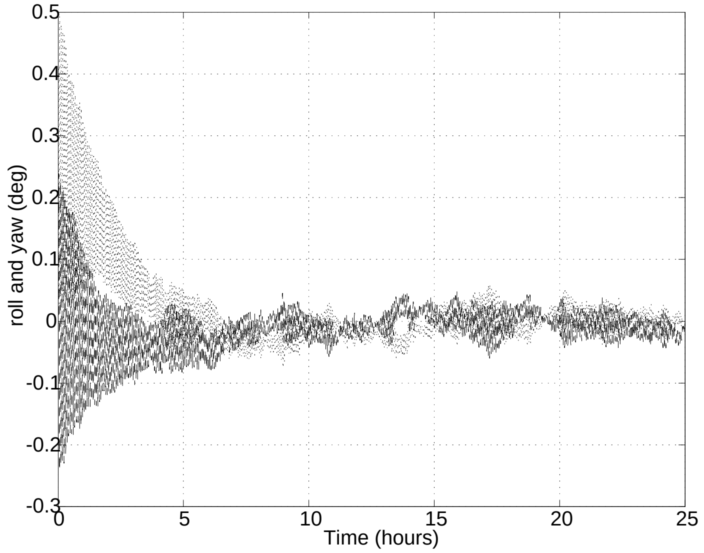

Fig. 3. Roll (dark) and Yaw (light) versus Time, LQG Control

equations of motion in state space form as

$$dx(t) = \begin{bmatrix} 0 & I \\ -M_s^{-1}K_s & -M_s^{-1}C_s \end{bmatrix} x(t)\,dt + \begin{bmatrix} 0 \\ M_s^{-1}B_s \end{bmatrix} u(t)\,dt + \begin{bmatrix} 0 \\ -\Gamma_s \end{bmatrix} dw(t)$$

where $M_s = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$, $B_s = \begin{bmatrix} -4k_c \cos\alpha \\ 0 \\ 0 \end{bmatrix}$,

$$C_s = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}, \quad \Gamma_s = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

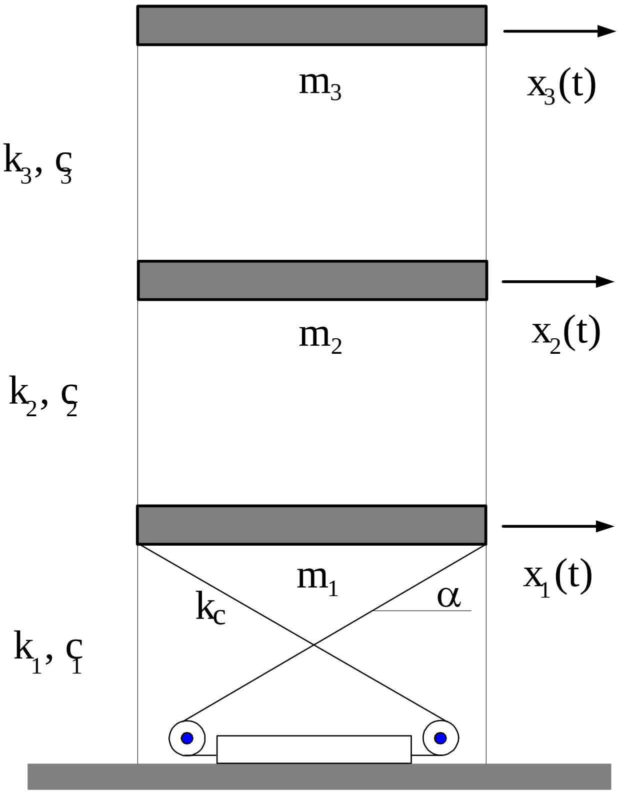

Fig. 4. Schematic Diagram for Three Degree-of-Freedom Structure

$$K_s = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix},$$ and m_i, c_i, k_i are the mass, damping, and stiffness, respectively, associated with the i-th floor of the building. And k_c is the stiffness of the tendon. The Brownian motion $w(t)$ with $E\{dw(t)\} = 0$ and $E\{dw(t)dw'(t)\} = Wdt$; in this example, $W = 1.00 \times 2\pi$ in^2/sec^3. The parameters were chosen to match modal frequencies and dampings of a experimental structure. The cost criterion is given by

$$J = \int_0^{t_F} \left(z'(t) K_s z(t) + k_c u^2(t) \right) \, dt$$

together with $R = k_c$, where z is a vector of floor displacements and $x = [z \ \dot{z}]'$.

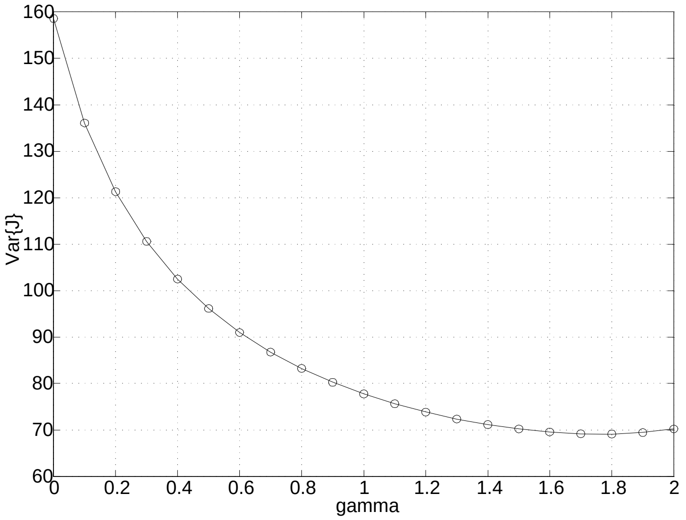

Fig. 5. Optimal Variance; Full-State-Feedback, MCV, 3DOF

Figure 5 shows that the variance of the cost criterion decreases as γ increases. Note that the $\gamma = 0$ point corresponds to the classical LQG case.

Figure 6 shows the RMS displacement responses of first (σ_{x_1}), second (σ_{x_2}), and third (σ_{x_3}) floor; and the RMS velocity responses of first (σ_{x_4}), second (σ_{x_5}), and third (σ_{x_6}) floor respectively, versus the MCV parameter, γ. It is important to note that both third floor RMS displacement and velocity responses can be decreased by choosing large γ.

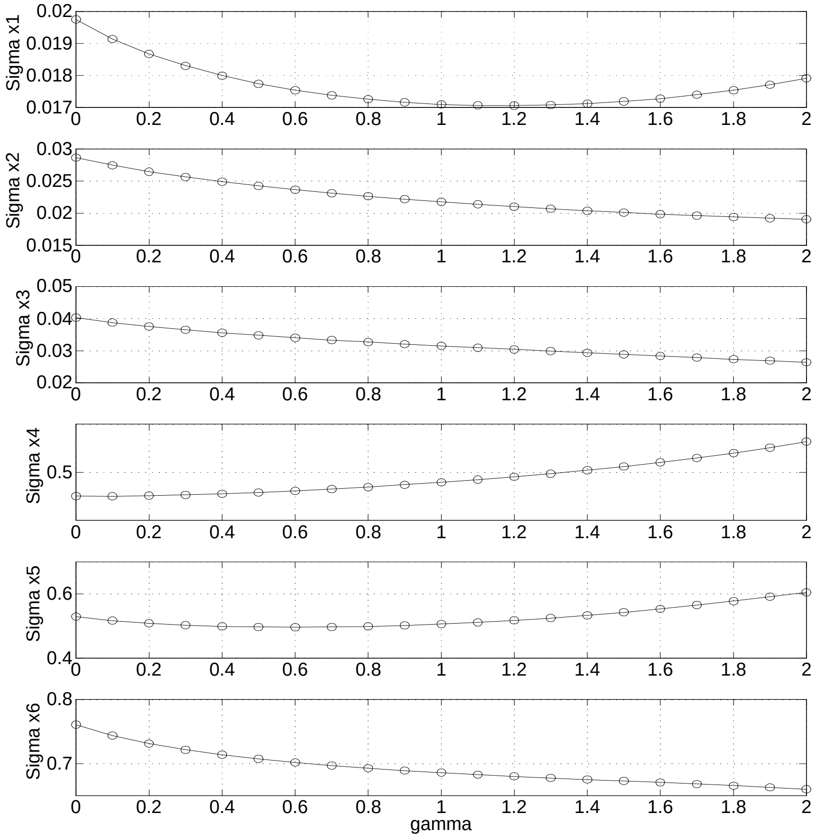

Fig. 6. Displacements and Velocities; Full-State-Feedback, MCV, 3DOF

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