

Discrete-Time, Bi-Cumulant Minimax and Nash Games

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Abstract—Continuous time, cumulant games have gathered interest, recently, but not much has been done for the discrete time case. In this paper, this problem will be addressed. Discrete time, two player, Nash and minimax cumulant games will be formulated and developed for a nonlinear system with non-quadratic costs. A recursion equation for the determining the equilibrium solutions is derived. For the linear quadratic case, equilibrium solutions are determined. Furthermore, through the use of cumulants, generalizations of H_2/H_∞ and H_∞ control are shown.

I. INTRODUCTION

Cumulants have gained interest recently for their use in stochastic optimal control, particularly for the continuous time case. Linear quadratic gaussian (LQG) control, itself can be considered a version of cumulant control in which only the first cumulant, the mean, of the cost is minimized. Another well known cumulant control technique is risk sensitive control, in which the performance index can be seen to be expanded to show that it minimizes a series of all the cost function cumulants. In fact, the cumulants are found through a series expansion of the second characteristic function, in this case, of the cost. Furthermore, if all of the cumulants are known, then one has a complete characterization of the distribution. This could be very useful in the area of control. By selecting and minimizing a linear combination of cumulants, one can affect the overall shape of the cost function's probability distribution. Fortunately, several control methods have begun to use this property. In [11], Sain, Won, et al., were able to formulate a minimal cost variance (MCV) control for continuous time. The work of Pham minimized a linear combination of k cost cumulants, also for the continuous time. The discrete time case grew from the work of Cosenza [2]. This work was then applied to a building vibration problem in [3].

While there has been some work with cumulants in control, there has been less so in the area of stochastic dynamical games. Game theory is an important extension of this work for many reasons, but one particular one is for its formulation of H_∞ and H_2/H_∞ control [1], [8]. In the case in which there is a stochastic process noise, the players' performance indices become the mean of the cost as opposed to the cost itself. With this knowledge, and the revelation that the mean is one of the cumulants, it is straightforward to think of extending these results for higher order cumulants. In the

continuous time, this has been done in [9] and [6]. This paper will examine it from the discrete time perspective.

In this paper, a cumulant based, discrete time, Nash game will be developed. At first we will consider a nonlinear system with a non-quadratic cost, and a dynamic programming approach will be taken to determine sufficient conditions for the equilibrium solution. The results from this framework will be applied to the case when the system is linear and the cost is quadratic. A Nash equilibrium solution will be determined. In this game the two players will be the control and a disturbance, in which the control wishes to minimize the cost variance of its cost, while the disturbance wishes to minimize the mean of its cost. When this is completed, attention will be turned to the case when both or more players are concerned with more than just the mean. A zero-sum game involving the cost variance will be done, at first from a nonlinear, non-quadratic framework; and then equilibrium solutions will be solved for the linear quadratic case.

II. TWO PLAYER NASH GAME

A. Problem Definition

The two player game will consider the nonlinear system

$$\begin{aligned} x(k+1) &= f(k, x(k), u(k), w(k)) + \sigma(k, x(k))\xi(k) \\ y(k) &= g(k, x(k), v(k)) \end{aligned} \quad (1)$$

where $x(0) = x_0$, $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the control, $w(k) \in \mathbb{R}^p$, $v(k) \in \mathbb{R}^r$ are disturbances, $y \in \mathbb{R}^q$ is the output, and $\xi(k) \in \mathbb{R}^d$ is the process noise. The index of times $k = 0, \dots, K$ is denoted as I and f, σ are assumed to be Borel measurable function. The process noise $\xi(k)$ is assumed to have a known probability distribution function.

The control u and disturbance w , have cost functions defined as

$$\begin{aligned} J_1(0, x(0); u, w) &= \sum_{k=0}^{K-1} L_1(k, x(k), u(k), w(k)) \\ &\quad + \psi_1(K, x(K)) \\ J_2(0, x(0); u, w) &= \sum_{k=0}^{K-1} L_2(k, x(k), u(k), w(k)) \\ &\quad + \psi_2(K, x(K)) \end{aligned} \quad (2)$$

where ψ_1, L_1, ψ_2 , and L_2 are nonnegative functions. In this Nash game, the control desires to minimize the variance, while the mean of its cost is constrained. The disturbance simply wishes to minimize the mean of its cost. This gives performance indices of

$$\begin{aligned}\phi_1(k, x, u, w) &= \text{Var}\{J_1(k, x(k); u, w)|Z(k)\} \\ \phi_2(k, x, u, w) &= E\{J_2(k, x(k); u, w)|Z_2(k)\}\end{aligned}\quad (3)$$

where ϕ_1 is the performance index for u , ϕ_2 is the performance index for w , and the mean of the control's cost is constrained to a function $M(k, Z(k))$. The information set at a time k for the control will be defined as $Z(k) = \{Z(k-1), y(k), u(k-1)\}$ where the initial data is set to be $Z(0) = \{y(0)\}$. A similar definition is made for the disturbance w and the information set is given as $Z_2(k) = \{Z_2(k-1), y(k), w(k-1)\}$, with $Z_2(0) = \{y(0)\}$. Feedback equilibrium strategies will be of interest to us in this problem, that is strategies of the form $u(k) = \mu(k, x(k))$ and $w(k) = \nu(k, x(k))$. The class of admissible feedback strategies μ and ν will be denoted by \mathcal{U}_F and \mathcal{W}_F respectively. The problem will be to determine a feedback Nash equilibrium solution for the game described by (1) and performance indices (3).

Definition 1: The pair (μ^*, ν^*) is a Nash equilibrium solution if it satisfies the inequalities

$$\begin{aligned}\phi_1(k, x, \mu^*, \nu^*) &\leq \phi_1(k, x, \mu, \nu^*) \\ \phi_2(k, x, \mu^*, \nu^*) &\leq \phi_2(k, x, \mu^*, \nu)\end{aligned}$$

for all $\mu \in \mathcal{U}_F$ and $\nu \in \mathcal{W}_F$.

B. Recursion Equations

In determining the equilibrium we can see that the problem is that of two minimization problems. Assume that ν^* has been played, and then minimize ϕ_1 . To achieve the first part of this equilibrium it is noted that minimizing the control's performance index can also be accomplished by solving the problem in which one minimizes the variance while constraining the mean. The problem then is to minimize

$$\text{Var}\{J_1|Z(0)\} = E\{J_1^2|Z(0)\} - E^2\{J_1|Z(0)\}$$

while satisfying

$$E\{J_1|Z(0)\} = M(0, Z(0))$$

where $M(k, Z(k))$ is the mean constraint function. To accomplish this task we first shall determine the one-step solution. Let $L_1(k) = L_1(k, x(k), \mu(k, x(k)), \nu^*(k, x(k)))$ and $\psi_1(K) = \psi_1(K, x(K))$ for notational purposes. Also define the variance cost as

$$V(k, Z(k)) := \text{Var}\{J_1(k, x(k); \mu^*, \nu^*)|Z(k)\}$$

in which μ^* is the one-step optimal strategy.

Using these substitutions and by expansion we have

$$\begin{aligned}V(K-1, Z(K-1)) &= \min_{\mu^{(K-1)}} \left\{ \right. \\ &E\{E\{\psi_1^2(K)|Z(K)\}|Z(K-1)\} \\ &+ 2E\{\psi_1(K)L_1(K-1)|Z(K-1)\} \\ &+ E\{L_1^2(K-1)|Z(K-1)\} - E^2\{\psi_1|Z(K-1)\} \\ &- 2E\{\psi_1(K)|Z(K-1)\}E\{L_1(K-1)|Z(K-1)\} \\ &- E^2\{L_1(K-1)|Z(K-1)\} \\ &+ 4\gamma(K-1)\left[E\{\psi_1(K) + L_1(K-1)|Z(K-1)\} \right. \\ &\left. \left. - M(K-1, Z(K-1))\right]\right\}\end{aligned}$$

where the nested expectations may be used if $Z(K-1) \subset Z(K)$, which by definition is true. By adding and subtracting $E\{E^2\{\psi_1(K, x(K))|Z(K)\}|Z(K-1)\}$, the one-step recursion equation can be further reduced to

$$\begin{aligned}V(K-1, Z(K-1)) &= \min_{\mu} \left\{ \right. \\ &E\{V(K, Z(K))|Z(K-1)\} \\ &+ E\{E^2\{\psi_1(K, x(K))|Z(K)\}|Z(K-1)\} \\ &+ E\{L_1^2(K-1)|Z(K-1)\} \\ &- E^2\{L_1(K-1)|Z(K-1)\} - E^2\{\psi_1|Z(K-1)\} \\ &+ 2E\{\psi_1(K)L_1(K-1)|Z(K-1)\} \\ &- 2E\{\psi_1(K)|Z(K-1)\}E\{L_1(K-1)|Z(K-1)\} \\ &+ 4\gamma(K-1)\left[E\{\psi_1(K) + L_1(K-1)|Z(K-1)\} \right. \\ &\left. \left. - M(K-1, Z(K-1))\right]\right\}\end{aligned}$$

where

$$V(K, Z(K)) = E\{\psi_1^2(K)|Z(K)\} - E^2\{\psi_1(K)|Z(K)\}.$$

With this one-step equation for the cost variance, induction may be used to find a more general cost variance recursion equation for a time $k \in I$.

Theorem 1: Consider the nonlinear two player game given in (1) and (3). The pair (μ^*, ν^*) is a Nash equilibrium strategy if there exist solutions $V(k, Z(k))$ and $P(k, Z(k))$ to

$$\begin{aligned}V(k, Z(k)) &= \min_{\mu^{(k)}} \left\{ E\{V(k+1, Z(k+1))|Z(k)\} \right. \\ &+ E\{L_1^2(k)|Z(k)\} - E^2\{L_1(k)|Z(k)\} \\ &+ E\left\{E^2\left\{\psi_1(K) + \sum_{j=k+1}^{K-1} L_1^*(j)\right|Z(k+1)\right\}\bigg|Z(k)\right\} \\ &+ E^2\left\{E\left\{\psi_1(K) + \sum_{j=k+1}^{K-1} L_1^*(j)\right|Z(k+1)\right\}\bigg|Z(k)\right\}\end{aligned}\quad (4)$$

$$\begin{aligned}
& + 2E \left\{ \left(\psi_1(K) + \sum_{j=k+1}^{K-1} L_1^*(j) \right) L_1(k) \middle| Z(k) \right\} \\
& - 2E \left\{ \psi_1(K) + \sum_{j=k+1}^{K-1} L_1^*(j) \middle| Z(k) \right\} E \{ L_1(k) | Z(k) \} \\
& + 4\gamma(k) \left[E \left\{ \psi_1(K) + \sum_{j=k+1}^{K-1} L_1^*(j) + L(k) \middle| Z(k) \right\} \right. \\
& \left. M(k, Z(k)) \right] \Big\}
\end{aligned}$$

and

$$\begin{aligned}
P(k, Z(k)) = \min_{\nu(k)} & \left\{ E \{ P(k+1, Z(k+1)) | Z(k) \} \right. \\
& \left. + E \{ L_2(k, x(k); \mu^*, \nu) | Z(k) \} \right\} \quad (5)
\end{aligned}$$

where $\gamma(k)$ is a Lagrange multiplier, $L_1^*(j) = L_1(j, x(j), \mu^*(j, x(k)), \nu^*(j, x(k)))$, and μ^* and ν^* are the minimizing arguments of (4) and (5) respectively.

Proof. Let the disturbance play its equilibrium solution. The results from the one-step analysis, shows that the variance cost is minimized. By the method of induction, (4) will now be shown to hold. Assume that (4) holds for time $k+1$, then it must be shown that it is valid for time k . By the definition of $V(k, Z(k))$ and letting $J_1(k) = J_1(k, x(k); \mu, \nu^*)$, we have

$$\begin{aligned}
V(k, Z(k)) = \min_{\mu(k), \dots, \mu(K-1)} & \left\{ E \{ J_1^2(k) | Z(k) \} \right. \\
& - E^2 \{ J_1(k) | Z(k) \} \\
& + 4\gamma(k) [E \{ J_1(k) | Z(k) \} \\
& \left. - M(k, Z(k))] \right\}
\end{aligned}$$

which by substitution gives

$$\begin{aligned}
V(k, Z(k)) = \min_{\mu(k), \dots, \mu(K-1)} & \left\{ E \{ L_1^2(k) | Z(k) \} \right. \\
& + 2E \{ L_1(k) J_1(k+1) | Z(k) \} \\
& + E \{ J_1^2(k+1) | Z(k) \} - E^2 \{ L_1(k) | Z(k) \} \\
& - E^2 \{ J_1(k+1) | Z(k) \} \\
& - 2E \{ L_1(k) | Z(k) \} E \{ J_1(k+1) | Z(k) \} \\
& \left. + 4\gamma(k) [E \{ J_1(k) | Z(k) \} - M(k, Z(k))] \right\}.
\end{aligned}$$

With the use of the principle of optimality we have

$$\begin{aligned}
V(k, Z(k)) = \min_{\mu(k)} & \left\{ E \{ L_1^2(k) | Z(k) \} \right. \\
& - E^2 \{ L_1(k) | Z(k) \} \\
& + 2E \left\{ (J_1^*(k+1)) L_1(k) \middle| Z(k) \right\} \\
& \left. - 2E \left\{ J_1^*(k+1) \middle| Z(k) \right\} E \{ L_1(k) | Z(k) \} \right.
\end{aligned}$$

$$\begin{aligned}
& + E \left\{ E \left\{ (J_1^*(k+1))^2 \middle| Z(k+1) \right\} \middle| Z(k) \right\} \\
& - E \left\{ E^2 \left\{ J_1^*(k+1) \middle| Z(k+1) \right\} \middle| Z(k) \right\} \\
& + E \left\{ E^2 \left\{ J_1^*(k+1) \middle| Z(k+1) \right\} \middle| Z(k) \right\} \\
& - E^2 \left\{ E \left\{ J_1^*(k+1) \middle| Z(k+1) \right\} \middle| Z(k) \right\} \\
& + 4\gamma(k) \left[E \left\{ J_1^*(k+1) + L(k) \middle| Z(k) \right\} \right. \\
& \left. - M(k, Z(k)) \right] \Big\}
\end{aligned}$$

where $J_1^*(k) = J_1(k, x(k); \mu^*, \nu^*)$. We now only have the mean constraint for time k to deal with. But for time $k+1$, the mean constraint is satisfied if the optimal solution $\mu^*(k+1, x(k+1))$ is played. Therefore equation (4) is satisfied for time k . For the disturbance ν , when the control plays its optimal solution, the problem becomes a stochastic LQR problem in which the disturbance wishes to minimize the mean of a cost. This problem gives the recursion equation in (5) \square

This result gives a sufficient condition for the Nash game strategies. To determine an important class of equilibrium solutions, we will consider the linear quadratic special case. Let the system be linear as described by

$$\begin{aligned}
x(k+1) = A(k)x(k) + B(k)u(k) \\
+ D(k)w(k) + E(k)\xi(k), \quad (6)
\end{aligned}$$

where A, B, D, E are, respectively, $n \times n, n \times m, n \times p, n \times d$ matrices. Also consider cost functions

$$\begin{aligned}
J_1 = x'(K)Q_K^1 x(K) + \sum_{k=0}^{K-1} & \left[x'(k)Q_1(k)x(k) \right. \\
& \left. + u'(k)R_{11}(k)u(k) + w'(k)R_{12}(k)w(k) \right] \\
J_2 = x'(K)Q_K^2 x(K) + \sum_{k=0}^{K-1} & \left[x'(k)Q_2(k)x(k) \right. \\
& \left. + u'(k)R_{21}(k)u(k) + w'(k)R_{22}(k)w(k) \right] \quad (7)
\end{aligned}$$

where R_{11} and R_{22} are positive definite. We will consider the full state feedback problem, therefore $y(k) = x(k)$. Furthermore, the process noise ξ is to be a white Gaussian discrete time random process with covariance

$$E \{ \xi(k) \xi'(k) \} = W(k)$$

and zero mean. Assume a form of the mean constraint to be $M(k, Z(k)) = x'(k)\mathcal{M}(k)x(k) + m(k)$ and, also,

that $V(k, Z(k)) = x'(k)\mathcal{V}(k)x(k) + v(k)$. Also, define

$$\begin{aligned}\Phi^1(k) &= I - D(k)(R_{22}(k) + D'(k)\mathcal{P}(k+1)D(k))^{-1} \\ &\quad \cdot D'(k)\mathcal{P}(k+1) \\ \Phi^2(k) &= I - B(k)(\gamma(k)R_{11}(k) + B'(k)\Lambda(k)B(k))^{-1} \\ &\quad \cdot B'(k)\Lambda(k) \\ \Lambda(k) &= \frac{1}{4}\mathcal{V}(k+1) + \gamma(k)\mathcal{M}(k+1) \\ &\quad + \mathcal{M}(k+1)E(k)W(k)E'(k)\mathcal{M}(k+1) \\ F(k) &= A(k) - B(k)K_1(k) - D(k)K_2(k)\end{aligned}\quad (8)$$

Corollary 1: Assume the conditions presented for the linear quadratic special case. Then, if the inverses exist, minimizing equation (4) with respect to μ gives

$$\begin{aligned}\mu^*(k, x(k)) &= -K_1(k)x(k) \\ &= [R_{11}(k) + B'(k)\Lambda(k)\Phi^1(k)B(k)]^{-1} \\ &\quad \cdot B'(k)\Lambda(k)\Phi^1(k)A(k) \\ \nu^*(k, x(k)) &= -K_2(k)x(k) \\ &= [R_{22}(k) + D'(k)\mathcal{P}(k+1)\Phi^2(k)D(k)]^{-1} \\ &\quad \cdot D'(k)\mathcal{P}(k+1)\Phi^2(k)A(k)\end{aligned}\quad (9)$$

where

$$\begin{aligned}\mathcal{M}(k) &= F'(k)\mathcal{M}(k+1)F(k) + Q_1(k) \\ &\quad + K_1'(k)R_{11}(k)K_1(k) + K_2'(k)R_{12}(k)K_2(k),\end{aligned}\quad (10)$$

$$\begin{aligned}\mathcal{V}(k) &= F'(k)\mathcal{V}(k+1)F(k) \\ &\quad + 4\mathcal{M}(k+1)E(k)W(k)E'(k)\mathcal{M}(k+1),\end{aligned}\quad (11)$$

$$\begin{aligned}\mathcal{P}(k) &= F'(k)\mathcal{P}(k+1)F(k) + Q_2(k) \\ &\quad + K_1'(k)R_{21}(k)K_1(k) \\ &\quad + K_2'(k)R_{22}(k)K_2(k),\end{aligned}\quad (12)$$

with $\mathcal{M}(K) = Q_K^1$, $\mathcal{V}(K) = 0$, and $\mathcal{P}(K) = Q_K^2$. Furthermore, we can construct

$$\begin{aligned}m(k) &= m(k+1) \\ &\quad + \text{tr}(E(k)W(k)E'(k)\mathcal{M}(k+1)),\end{aligned}\quad (13)$$

$$\begin{aligned}v(k) &= v(k+1) \\ &\quad + \text{tr}(E(k)W(k)E'(k)\mathcal{V}(k+1)) \\ &\quad + E\{(\xi'(k)E'(k)\mathcal{M}(k+1)E(k)\xi(k))^2|Z(k)\} \\ &\quad - \text{tr}^2(E(k)W(k)E'(k)\mathcal{M}(k+1)),\end{aligned}\quad (14)$$

$$p(k) = p(k+1) + \text{tr}(E(k)W(k)E'(k)\mathcal{P}(k+1)) \quad (15)$$

where $m(K) = 0$, $v(K) = 0$, and $p(K) = 0$.

Proof. Due to space constraints, the proof is omitted. For details, see [7]. \square

C. Relation to H_2/H_∞ and Multi-Objective Cumulant Control for Discrete-Time Systems

With the results from the linear quadratic special case, it is possible to present a multi-objective cumulant control method, but first a relation to what is meant by multi-objective must be given. For the purposes of this dissertation, we will mainly be referring to H_2/H_∞ control. In this problem, for stochastic systems, one wishes to minimize the H_2 norm of the system while constraining the H_∞ norm, which corresponds to a game in which the control minimizes the mean of a specific cost J_1 and the disturbance minimizing the mean of a cost J_2 . Both of these costs are specific cost functions, that will be defined shortly. The system will still be a linear one describe by (6). In addition to the linear system, allow the regulated outputs of the system be given as

$$\begin{aligned}z_1(k) &= G_1(k)x(k) + H_1(k)u(k) \\ z_2(k) &= G_2(k)x(k) + H_2(k)u(k)\end{aligned}$$

where $G_1'G_1 = Q_1$, $H_1'H_1 = R_{11}$, $G_1'H_1 = 0$ and $G_2'G_2 = Q_2$, $H_2'H_2 = R_{12}$, $G_2'H_1 = 0$. Also, let $Q_K^1 = Q_K^2 = 0$. The cost functions are specified as

$$\begin{aligned}J_1 &= \sum_{k=0}^{K-1} z_1'(k)z_1(k) \\ J_2 &= \sum_{k=0}^{K-1} \delta^2 w'(k)w(k) - z_2'(k)z_2(k)\end{aligned}\quad (16)$$

where $\delta > 0$ is a constant. The H_∞ norm from w to z_2 is

$$\|T_{zw}\|_\infty = \sup_{w \in \mathcal{W}} \frac{\|z_2\|}{\|w\|}$$

where the norm $\|\cdot\|$ for a function u is defined as

$$\|u\|^2 = E \left\{ \sum_{k=0}^{K-1} u'(k)u(k) \right\}.$$

Consider the sub-optimal problem in which the we wish the H_∞ norm to be less than some constant, δ . Then what we have is

$$\|T_{zw}\|_\infty = \sup_{w \in \mathcal{W}} \frac{\|z_2\|}{\|w\|} < \delta$$

which leads to

$$\|z_2\|^2 < \delta^2 E \left\{ \sum_{k=0}^{K-1} w'(k)w(k) \right\}.$$

But this is simply

$$E \left\{ \sum_{k=0}^{K-1} [\delta^2 w'(k)w(k) - z_2'(k)z_2(k)] \right\} > 0$$

and notice that, assuming that $E\{J_2\} > 0$, the above problem amounts to minimizing the mean of J_2 with respect to the disturbance. For the H_2/H_∞ problem, the goal would be to determine an equilibrium solution of the game presented, with the control's optimal strategy being its H_2/H_∞ control law. Notice that if we use the non-zero sum cost variance game, we can have the situation in which the disturbance still wishes to minimize the mean of its cost function, J_2 (thereby constraining the H_∞ norm of the system to a constant δ), while we can use a different performance index for the control, namely the variance. This creates the discrete time, multi-objective cumulant control law, and it can be seen as a generalization of the H_2/H_∞ control problem, in the sense that control's performance index goes beyond the first cumulant. To accomplish this, the cost functions given in (16), along with the previous result (9), can be used. We see that the control and disturbance gain matrices are given as

$$\begin{aligned} K_1(k) &= [\gamma(k)R_1(k) + B'(k)\Lambda(k)\Phi^1(k)B(k)]^{-1} \\ &\quad \cdot B'(k)\Lambda(k)\Phi^1(k)A(k) \\ K_2(k) &= [\delta^2 + D'(k)\mathcal{P}(k+1)\Phi^2(k)D(k)]^{-1} \\ &\quad \cdot D'(k)\mathcal{P}(k+1)\Phi^2(k)A(k) \end{aligned} \quad (17)$$

with

$$\begin{aligned} \Phi^1(k) &= I - D(k) (\delta^2 + D'(k)\mathcal{P}(k+1)D(k))^{-1} \\ &\quad \cdot D'(k)\mathcal{P}(k+1) \\ \Phi^2(k) &= I - B(k) (\gamma(k)R_1(k) + B'(k)\Lambda(k)B(k))^{-1} \\ &\quad \cdot B'(k)\Lambda(k) \end{aligned} \quad (18)$$

through substitution. Defining $F(k) = A(k) - B(k)K_1(k) - D(k)K_2(k)$, we then have

$$\begin{aligned} \mathcal{M}(k) &= F'(k)\mathcal{M}(k+1)F(k) + Q_1(k) \\ &\quad + K_1'(k)R_1(k)K_1(k) \\ \mathcal{V}(k) &= F'(k)\mathcal{V}(k+1)F(k) \\ &\quad + 4\mathcal{M}(k+1)E(k)W(k)E'(k)\mathcal{M}(k+1) \\ \mathcal{P}(k) &= F'(k)\mathcal{P}(k+1)F(k) + \delta^2 K_2'(k)D(k)K_2(k) \\ &\quad - K_1'(k)R_2(k)K_1(k) - Q_2(k) \end{aligned} \quad (19)$$

where $\mathcal{P}(K) = 0$, $\mathcal{M}(K) = \mathcal{V}(K) = 0$. This gives a method to find a control law that accounts for both a higher order cumulant, while allowing room to design for some uncertainty through the H_∞ norm.

III. ZERO-SUM COST VARIANCE GAME

So far the problem has been a game in which the control u wished to minimize the cost variance, while the disturbance w wished to minimize the mean of its cost. This Nash game was developed for the nonlinear system with a non-quadratic cost and then applied to

the linear quadratic case. In this section we will do the same, except the two players will both have the same performance index, its cost variance, and the control will wish to minimize the index, whereas the disturbance will seek to be a maximizer, while the mean is constrained.

A. Recursion Equation for Nonlinear Systems with Non-Quadratic Costs

The system under consideration is exactly the same as before and is described by (1). The cost function is given as

$$\begin{aligned} J(0, x(0); u, w) &= \sum_{k=0}^{K-1} L(k, x(k), u(k), w(k)) \\ &\quad + \psi(K, x(K)) \end{aligned} \quad (20)$$

where L and ψ are non-negative functions. Also, we will let $Z(k) = \{Z(k-1), u(k-1), w(k-1)\}$ be the information available to the players. In the problem presented here, the first player, the control u , wishes to minimize a linear combination of the mean and variance of J , while the disturbance wishes to maximize this same quantity.

Definition 2: The pair (μ^*, ν^*) is the zero-sum (or minimax) equilibrium solution if it satisfies the inequality

$$\phi(k, x, \mu^*, \nu) \leq \phi(k, x, \mu^*, \nu^*) \leq \phi(k, x, \mu, \nu^*)$$

for all $\mu \in \mathcal{U}_F$ and $\nu \in \mathcal{W}_F$.

A sufficient condition is given by the following theorem.

Theorem 2: Consider the nonlinear two player game given in (1) and (20). A solution pair (μ^*, ν^*) is a zero-sum equilibrium strategy if there exists a solution $V(k, Z(k))$ to

$$\begin{aligned} V(k, Z(k)) &= \min_{\mu(k)} \max_{\nu(k)} \left\{ E\{V(k+1, Z(k+1)) | Z(k)\} \right. \\ &\quad \left. + E\{L_1^2(k) | Z(k)\} - E^2\{L_1(k) | Z(k)\} \right. \\ &\quad \left. + E\left\{ E^2\left\{ \psi_1(K) + \sum_{j=k+1}^{K-1} L_1^*(j) \middle| Z(k+1) \right\} \middle| Z(k) \right\} \right. \\ &\quad \left. + E^2\left\{ E\left\{ \psi_1(K) + \sum_{j=k+1}^{K-1} L_1^*(j) \middle| Z(k+1) \right\} \middle| Z(k) \right\} \right. \\ &\quad \left. + 2E\left\{ \left(\psi_1(K) + \sum_{j=k+1}^{K-1} L_1^*(j) \right) L_1(k) \middle| Z(k) \right\} \right. \\ &\quad \left. - 2E\left\{ \psi_1(K) + \sum_{j=k+1}^{K-1} L_1^*(j) \middle| Z(k) \right\} E\{L_1(k) | Z(k)\} \right. \\ &\quad \left. + 4\gamma(k) \left[E\left\{ \psi_1(K) + \sum_{j=k+1}^{K-1} L_1^*(j) + L(k) \middle| Z(k) \right\} \right] \right\} \end{aligned}$$

$$\left. \begin{aligned} & - M(k, Z(k)) \end{aligned} \right\} \quad (21)$$

and

$$\begin{aligned} V(k, Z(k)) = & \max_{\nu^{(k)}} \min_{\mu^{(k)}} \left\{ E\{V(k+1, Z(k+1))|Z(k)\} \right. \\ & + E\{L_1^2(k)|Z(k)\} - E^2\{L_1(k)|Z(k)\} \\ & + E\left\{ E^2\left\{ \psi_1(K) + \sum_{j=k+1}^{K-1} L_1^*(j) \middle| Z(k+1) \right\} \middle| Z(k) \right\} \\ & + E^2\left\{ E\left\{ \psi_1(K) + \sum_{j=k+1}^{K-1} L_1^*(j) \middle| Z(k+1) \right\} \middle| Z(k) \right\} \\ & + 2E\left\{ \left(\psi_1(K) + \sum_{j=k+1}^{K-1} L_1^*(j) \right) L_1(k) \middle| Z(k) \right\} \\ & - 2E\left\{ \psi_1(K) + \sum_{j=k+1}^{K-1} L_1^*(j) \middle| Z(k) \right\} E\{L_1(k)|Z(k)\} \\ & + 4\gamma(k) \left[E\left\{ \psi_1(K) + \sum_{j=k+1}^{K-1} L_1^*(j) + L(k) \middle| Z(k) \right\} \right. \\ & \left. - M(k, Z(k)) \right\} \end{aligned} \quad (22)$$

where $\gamma(k)$ is a Lagrange multiplier, $L_1^*(j) = L_1(j, x(j), \mu^*(j, x(k)), \nu^*(j, x(k)))$, and μ^* and ν^* are the minimizing and maximizing arguments of (21) and (22) respectively.

Proof.

Assume that the recursion equations in the theorem are true for time $k+1$; we will show that they then hold for time k . Assume that the control has played its equilibrium solution μ^* . Then for (22), we have

$$\begin{aligned} V(k, Z(k)) = & \max_{\nu^{(k)}, \dots, \nu^{(K-1)}} \left\{ \right. \\ & E\{J^2(k, x(k); \mu^*, \nu)|Z(k)\} \\ & - E^2\{J(k, x(k); \mu^*, \nu)|Z(k)\} \\ & + 4\gamma(k) [E\{J(k, x(k); \mu^*, \nu)|Z(k)\} \\ & \left. - M(k, Z(k))] \right\} \end{aligned} \quad (23)$$

which by substitution gives

$$\begin{aligned} V(k, Z(k)) = & \max_{\nu^{(k)}, \dots, \nu^{(K-1)}} \left\{ \right. \\ & E\{(L(k) + J(k+1))^2|Z(k)\} \\ & - E^2\{L(k) + J(k)|Z(k)\} \\ & \left. + 4\gamma(k) [E\{J(k+1)|Z(k)\} \right. \end{aligned}$$

$$\begin{aligned} & \left. - M(k, Z(k))\right\} \\ = & \max_{\nu^{(k)}, \dots, \nu^{(K-1)}} \left\{ E\{L^2(k)|Z(k)\} \right. \\ & + 2E\{L(k)J(k+1)|Z(k)\} \\ & + E\{J^2(k+1)|Z(k)\} - E^2\{L(k)|Z(k)\} \\ & - E^2\{J(k+1)|Z(k)\} \\ & - 2E\{L(k)|Z(k)\}E\{J(k+1)|Z(k)\} \\ & \left. + 4\gamma(k) [E\{J(k)|Z(k)\} - M(k, Z(k))] \right\} \end{aligned}$$

where J is given in (20) with k in place of 0 and $L(k)$ in this case is $L(k, x; \mu^*, \nu)$. Similarly, let $L^*(j) = L(j, x; \mu^*, \nu^*)$. By using the principle of optimality we have

$$\begin{aligned} V(k, Z(k)) = & \max_{\nu^{(k)}} \left\{ E\{L^2(k)|Z(k)\} \right. \\ & - E^2\{L(k)|Z(k)\} \\ & + 2E\{(J^*(k+1) L(k)|Z(k)\} \\ & - 2E\{J^*(k+1)|Z(k)\}L(k)|Z(k)\} \\ & + E\left\{ E\left\{ (J^*(k+1))^2|Z(k+1) \right\} \middle| Z(k) \right\} \\ & - E\left\{ E^2\{J^*(k+1)|Z(k+1)\} \middle| Z(k) \right\} \\ & + E\left\{ E^2\{J^*(k+1)|Z(k+1)\} \middle| Z(k) \right\} \\ & - E^2\left\{ E\{J^*(k+1)|Z(k+1)\} \middle| Z(k) \right\} \\ & + 4\gamma(k) [E\{J^*(k+1) + L(k)|Z(k)\} \\ & \left. - M(k, Z(k))] \right\} \end{aligned}$$

where we still only have the mean constraint for time k . But for time $k+1$, the mean constraint is satisfied if the optimal solution $\mu^*(k+1, x(k+1))$ is played. Therefore equation (22) is satisfied for time k . What is left to show is (21). Suppose that instead of μ^* being played in (23), ν^* is played. If we minimize, instead of maximize, then the proof for (22) can be followed to show (21). \square

As in the non-zero sum, two-player game, the system will be described by (6). Also, the cost is given by

$$\begin{aligned} J = & x'(K)Q_K x(K) + \sum_{k=0}^{K-1} \left[x'(k)Q(k)x(k) \right. \\ & \left. + u'(k)R(k)u(k) + w'(k)S(k)w(k) \right] \end{aligned} \quad (24)$$

where R and S are positive definite. Again, the full state feedback problem is under consideration and the process noise is defined in the same way as in the linear quadratic Nash game.

Assume $M(k, x(k)) = x'(k)\mathcal{M}(k)x(k) + m(k)$, $V(k, x(k)) = x'(k)\mathcal{V}(k)x(k) + v(k)$, as well as define

$$\begin{aligned}\Lambda(k) &= \frac{1}{4}\mathcal{V}(k+1) + \gamma(k)\mathcal{M}(k+1) \\ &\quad + \mathcal{M}(k+1)E(k)W(k)E'(k)\mathcal{M}(k+1), \\ F(k) &= A(k) - B(k)K_1(k) - D(k)K_2(k).\end{aligned}$$

Corollary 2: Assuming the linear quadratic case, optimizing (21) yields the equilibrium solution. Through substitution we get the equilibrium strategies

$$\begin{aligned}\mu^*(k, x(k)) &= -K_1(k)x(k) \\ &= -[R(k) + B'(k)\Lambda(k)\Phi^1(k)B(k)]^{-1} \\ &\quad \cdot B'(k)\Lambda(k)\Phi^1(k)A(k)x(k) \\ \nu^*(k, x(k)) &= -K_2(k)x(k) \\ &= -[S(k) + D'(k)\Lambda(k)\Phi^2(k)D(k)]^{-1} \\ &\quad \cdot D'(k)\Lambda(k)\Phi^2(k)A(k)x(k)\end{aligned}\tag{25}$$

where

$$\begin{aligned}\Phi^1(k) &= I - D(k)(\gamma(k)S(k) + D'(k)\Lambda(k)D(k))^{-1} \\ &\quad \cdot D'(k)\Lambda(k) \\ \Phi^2(k) &= I - B(k)(\gamma(k)R(k) + B'(k)\Lambda(k)B(k))^{-1} \\ &\quad \cdot B'(k)\Lambda(k)\end{aligned}\tag{26}$$

if the inverses exist. The equations for \mathcal{M} , \mathcal{V} , and \mathcal{P} are then given as

$$\begin{aligned}\mathcal{M}(k) &= F'(k)\mathcal{M}(k+1)F(k) + Q(k) \\ &\quad + K_1'(k)R(k)K_1(k) \\ &\quad + K_2'(k)S(k)K_2(k),\end{aligned}\tag{27}$$

and

$$\begin{aligned}\mathcal{V}(k) &= F'(k)\mathcal{V}(k+1)F(k) \\ &\quad + 4\mathcal{M}(k+1)E(k)W(k)E'(k)\mathcal{M}(k+1),\end{aligned}\tag{28}$$

where $\mathcal{M}(K) = Q_K$ and $\mathcal{V}(K) = 0$. If these solutions exist, then we have

$$\begin{aligned}m(k) &= m(k+1) \\ &\quad + \text{tr}(E(k)W(k)E'(k)\mathcal{M}(k+1))\end{aligned}\tag{29}$$

and

$$\begin{aligned}v(k) &= v(k+1) + \text{tr}(E(k)W(k)E'(k)\mathcal{V}(k+1)) \\ &\quad + E\{(\xi'(k)E'(k)\mathcal{M}(k+1)E(k)\xi(k))^2|Z(k)\} \\ &\quad - \text{tr}^2(E(k)W(k)E(k)\mathcal{M}(k+1)),\end{aligned}\tag{30}$$

where $m(K) = 0$ and $v(K) = 0$.

Proof. For the proof, see [7]. \square

B. Relationship with H_∞ Control

In the two player, Nash game discussion, a generalization, based upon cumulants, was given for H_2/H_∞ control. Likewise for the minimax game, a generalization of H_∞ control may be found using cumulants. This generalization stems from the work of Basar and Bernhard, [1]. In their formulation, only deterministic systems were considered, however, they also hinted that for stochastic systems that consider the mean of a cost, there is noise insensitivity. Noise insensitivity is simply meant to say that the equilibrium strategies are the same for both the deterministic and stochastic problems. To see that this is indeed the case for H_∞ problem, define the two norm in same way as before as

$$\|z\|_{2,[0,K]}^2 = \sum_{k=0}^{K-1} E\{z'(k)z(k)\}$$

for a signal z . Now let z be the regulated output of the system (6), given as

$$z(t) = G(t)x(t) + H(t)u(t).$$

The performance index is then given as

$$\begin{aligned}E\{J\} &= E\left\{\sum_{k=0}^{K-1} z'(k)z(k) - \delta w'(k)w(k)\right\} \\ &= \sum_{k=0}^{K-1} E\{z'(k)z(k)\} - \delta \sum_{k=0}^{K-1} E\{w'(k)w(k)\} \\ &= \|z\|_{2,[0,K]}^2 - \delta^2 \|w\|_{2,[0,K]}^2.\end{aligned}$$

Along with the two norm, we have the induced H_∞ norm

$$\|T_{zw}\|_\infty := \sup_{w \in \mathcal{W}} \frac{\|z\|_{2,[0,K]}}{\|w\|_{2,[0,K]}}$$

for $w \neq 0$. Let

$$\delta^* := \inf_{\mu \in \mathcal{U}} \|T_{z\mu}\|_\infty = \inf_{\mu \in \mathcal{U}} \sup_{w \in \mathcal{W}} \frac{\|z\|_{2,[0,K]}}{\|w\|_{2,[0,K]}}$$

so if μ^* is the control law satisfying δ^* , then

$$\|z_{\mu^*}\|_{2,[0,K]} \leq \delta^* \|w\|_{2,[0,K]}\tag{31}$$

for $w \in \mathcal{W}$ and where $z_{\mu^*} = Cx + D\mu^*$. As seen in [1], pp. 27-28, zero is an upper bound on $E\{J\}$ and furthermore in the case of saddlepoint solutions, the sup inf problem arrives at the same solution. Notice that the inequality (31) is equivalent to solving

$$\|z_{\mu^*}\|_{2,[0,K]}^2 - \delta^{*2} \|w\|_{2,[0,K]}^2 \leq 0.$$

Because this is simply an upper bound on $E\{J\}$ of zero, we can generalize the H_∞ control problem, with a stochastic perturbation, by extending our performance index from

$$\phi(k, x, u, w) = E\{J(k, x(k); u, w)|Z(k)\}$$

to

$$\begin{aligned} \phi(k, x, u, w) = & \text{Var}\{J(k, x(k); u, w)|Z(k)\} \\ & + \gamma E\{J(k, x(k); u, w)|Z(k)\}. \end{aligned}$$

Notice that by the use of this performance index, we can use the results from the linear quadratic subsection to determine a minimal variance H_∞ control strategy. That strategy is

$$\begin{aligned} \mu^*(k, x(k)) = & -K_1(k)x(k) \\ = & -[R(k) + B'(k)\Lambda(k)\Phi^1(k)B(k)]^{-1} \\ & \cdot B'(k)\Lambda(k)\Phi^1(k)A(k)x(k) \\ \nu^*(k, x(k)) = & -K_2(k)x(k) \\ = & -[S(k) + D'(k)\Lambda(k)\Phi^2(k)D(k)]^{-1} \\ & \cdot D'(k)\Lambda(k)\Phi^2(k)A(k)x(k) \end{aligned} \quad (32)$$

where

$$\begin{aligned} \Phi^1(k) = & I - D(k)(\gamma(k)S(k) + D'(k)\Lambda(k)D(k))^{-1} \\ & \cdot D'(k)\Lambda(k) \\ \Phi^2(k) = & I - B(k)(\gamma(k)R(k) + B'(k)\Lambda(k)B(k))^{-1} \\ & \cdot B'(k)\Lambda(k) \end{aligned} \quad (33)$$

$$\begin{aligned} \mathcal{M}(k) = & F'(k)\mathcal{M}(k+1)F(k) + Q(k) \\ & + K_1'(k)R(k)K_1(k) - \delta^2 K_2'K_2 \end{aligned} \quad (34)$$

$$\begin{aligned} \mathcal{V}(k) = & F'(k)\mathcal{V}(k+1)F(k) \\ & + 4\mathcal{M}(k+1)E(k)W(k)E'(k)\mathcal{M}(k+1) \end{aligned} \quad (35)$$

where $F(k) = A(k) - B(k)K_1(k) - D(k)K_2(k)$, $\mathcal{M}(K) = Q_K$, $\mathcal{V}(K) = 0$, and we assume that $C'C = Q$, $C'D = 0$, and $D'D = R$.

IV. CONCLUSION

Discrete time, cumulant Games have been formulated and developed for a class on nonlinear systems with non-quadratic costs. For the linear quadratic case, equilibrium solutions have been determined, along with their discrete time Riccati equations. Furthermore a generalization of the H_2/H_∞ control method has been discussed.

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