CUMULANTS AND RISK-SENSITIVE CONTROL: A COST MEAN AND VARIANCE THEORY WITH APPLICATION TO SEISMIC PROTECTION OF STRUCTURES *

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Abstract

The risk-sensitive optimal stochastic control problem has an interpretation in terms of managing the value of linear combinations of the cumulants of a traditional performance index. The coefficients in these linear combinations are fixed, explicit functions of the risk parameter. This paper demonstrates the possibility of controlling linear combinations of index cumulants with broader opportunities to choose the coefficients. In view of the considerable interest given to cumulants in the theories of signal processing, detection, and estimation over the last decade, such an interpretation offers the possibility of new insights into the broad modern convergence of the concepts of robust control in general. Considered in detail are the foundations for a full-state-feedback solution to the problem of controlling the second cumulant of a cost function, given modest constraints on the first cumulant. The formulation is carried out for a class of nonlinear stochastic differential equations, associated with an appropriate class of non-quadratic performance indices. A Hamilton-Jacobi framework is adopted; and the defining equations for solving the linear, quadratic case are

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determined. The method is then applied to a situation in which a building is to be protected from earthquakes. Densities of the cost function are computed, so as to give insight into the question of how the first and second cumulants affect a cost considered as a random variable.

1 Introduction

Relationships among different areas of robust control, such as H_∞ optimal control, game theory, and risk-sensitive (RS) stochastic control have been the subject of recent research. See, for example, references [10] and [19]. Because they represent differing paradigms for thinking about the topic, and because these varying paradigms may fit one application area better than another, the overall investigative effort will certainly profit from such studies. By way of illustration, Figure 1 gives a partial overview of some of the connections between the different areas of robust control. The purpose of the present paper is not to develop these interrelationships, *per se*, but rather to point out the rather natural possibility of expanding the notions involved with the RS body of thought.

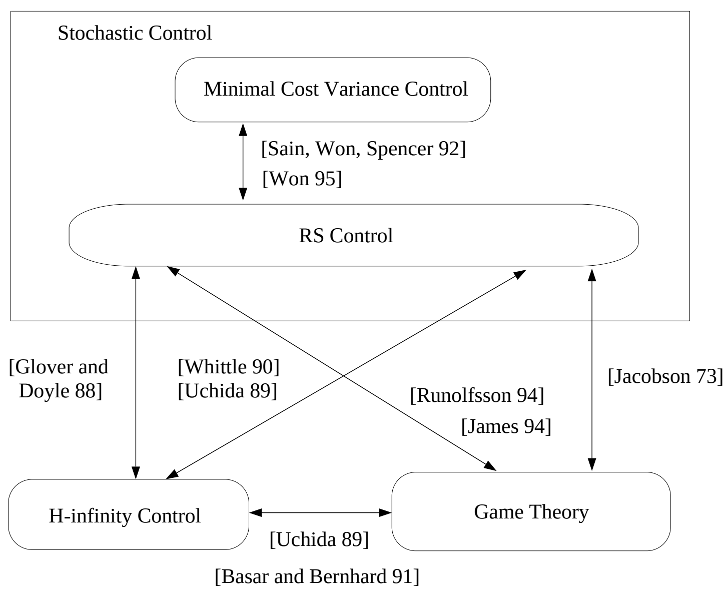

Figure 1: Relations Between Various Robust Controls

Consider an RS cost function [30], $J_{RS} = -\frac{2}{\theta}\log\left(E\left\{\exp\left(-\frac{\theta}{2}J\right)\right\}\right)$, where θ is a real parameter and J is given by

$$J(t, x(t), k) = \int_t^{t_F} \left[L(s, x(s), k(s, x(s)))\right] ds + \psi(x(t_F)), \qquad (1)$$

where $L : [t_0, t_F] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^+$ and $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^+$. Here we proceed intuitively, regarding $x(t)$ as a suitable random process, and $k(t, x(t))$ as a suitable feedback control mapping defined upon it. The moment generating function, or the first characteristic function of J, is $\phi_1(s) = E\{\exp(-sJ)\}$, and the cumulant generating function, or second characteristic function, follows by $\phi_2(s) = \log \phi_1(s)$. Suppose, for ease of conversation, that we may assume the existence of the cumulants associated with J. Then we have that $\phi_2(s) = \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \beta_i(J) s^i$, in which $\beta_i(J)$ is the ith *cumulant*, sometimes also called *semi-invariant*, of J. We remark that, if not all the cumulants exist, a truncated version, with remainder, can be written. It follows that

$$J_{RS} = \left(-\frac{2}{\theta}\right) \left\{ \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \beta_i(J) \left(\frac{\theta}{2}\right)^i \right\}. \tag{2}$$

Approximating to the second order,

$$\begin{aligned} J_{RS} &= \beta_1(J) - \frac{\theta}{4}\beta_2(J) + O(\theta^2) \\ &= E\{J\} - \frac{\theta}{4} VAR\{J\} + O(\theta^2). \end{aligned} \tag{3}$$

For small θ, then, we see that the RS optimization problem leads to the well understood task of optimizing the mean value $E\{J\}$ of J. If θ is not quite that small, then the next term to come into play involves $VAR\{J\}$. In this situation we have in essence a weighted linear combination of $E\{J\}$ and $VAR\{J\}$, which are in fact the first two cumulants of J. One can then address the optimization of the second cumulant $VAR\{J\}$, under the restriction that the first cumulant $E\{J\}$ exists. We call this the minimal cost variance (MCV) control problem.

Therefore, in accordance with the number of cumulants which exist, RS problems associate naturally with a linear combination of cost cumulants. The particular coefficients which appear in the linear combination are of course fixed. Our purpose in this paper is to show that linear combinations of cumulants can be addressed directly, in their own right, without such restrictions on coefficients. We believe that this is quite in tune with the original motivations of the risk-sensitive idea; and so we regard the present investigation as part of the broad notion of RS technique, though it suggests perhaps a broadening of the class of problems which have been studied thus far. This way of proceeding might be given the name *cost cumulant control*. Linear Quadratic Gaussian (LQG) control, MCV control, and RS control certainly can be grouped into this paradigm. Investigating such a notion is clearly the work of multiple persons over extended time. For the present instance, we focus on the MCV problem.

Minimal cost variance (MCV) control was first examined for the open-loop situation by Sain in a dissertation [20] at the University of Illinois in 1965. Certain of those ideas appeared in journal form in 1966 [21]. In 1971, Sain and Liberty published an open loop result on minimizing the performance variance while keeping the performance mean to a prespecified value [23]. In that paper, new mathematical representations were obtained, and for the first time analytical procedures were used to produce and display the cost densities associated with such control laws, as well as their effects upon the state and control variables of the system. Liberty continued to study characteristic functions of integral quadratic forms, further developing the MCV control idea. Some years later, with Hartwig, he published the results of generating cumulants in the time domain [16]. In 1992, Sain, Won, and Spencer showed that MCV control is related to RS control[24]. Cumulant control can also be viewed as a cost distribution shaping method. See [25, page 358] for the RS case. In classical LQG control, only one cumulant is controlled, but in MCV control there are two cumulants, namely the mean and the variance, of the cost function that we can control. Thus, one has extra design freedom to shape the cost distribution. This point is demonstrated in Section 7 of the sequel.

It is interesting to compare the time line of MCV, cost cumulants, and related topics, with that of RS optimization. Risk-sensitive optimal control seems to have started with Jacobson in 1973. In the 1970s Jacobson extended LQG results by replacing the quadratic criterion with the exponential of a quadratic cost functional, and related linear-exponential-quadratic-Gaussian (LEQG) control to differential games [12]. Many years later, Whittle noted Jacobson's results as an instance of RS control [31]. Speyer *et al.* [25] extended Jacobson's results to the noisy linear measurements case in discrete time. In [25], optimal control becomes a linear function of the smoothed history of the state, and the solutions are acquired by defining an enlarged state space composed of the entire state history. This enlarged state vector grows at every new stage but retains the feature of being a discrete linear system with additive white Gaussian noise. They also briefly discuss the continuous time terminal LEQG problem, and the solutions are achieved by taking a formal limit of the discrete case solutions. In 1976, Speyer considered the noisy measurement case again in continuous time, but with zero state weighting in the cost function [26]. Unlike the previous work [25], the Hamilton-Jacobi-Bellman equation was used to produce the solutions. Kumar and van Schuppen derived the general solution of the partially observed exponential-of-integral (EOI) problem in continuous time with zero plant noise in 1981 [15]. Whittle then published his results for the general solution of the partially observed logarithmic-exponential-of-integral (LEOI) problem in discrete time [29]. Four years later, Bensoussan and van Schuppen reported the solution to the general case of a continuous time partially observed stochastic EOI problem using a different method from Whittle [2]. In 1988, Glover and Doyle related H_∞ and minimum entropy criteria to the infinite horizon version of LEOI theory in discrete time, thus establishing a relationship between RS control and H_∞ optimal control [10]. This result was extended to continuous time by Glover [11]. In 1990, Whittle published the risk-sensitive maximum principle in book form [30], and published a journal article about the risk-sensitive maximum principle for the case of partially observed states using large deviation theory [31]. A year later, Bensoussan published a book with all solutions (including the partially observed case) of the exponential-of-integral problem [3]. Başar and Bernhard noted the relationship between deterministic dynamic games and H_∞ optimal control in their book [1]. In 1992, James states that the RS optimal control problem with full-state-feedback information is equivalent to a stochastic differential game problem [13]. Fleming and McEneaney independently pointed out similar results in [8]. More recently, in 1994, James, *et al.* published RS control and dynamic games solutions for partially observed discrete-time nonlinear systems [14]. Finally, Runolfsson presented the relationship between RS control and stochastic differential games in the infinite-horizon case using large deviation ideas [19]. The reader should have no difficulty with further following of this line of work, as it remains active today in the major journals and conferences.

Viewed, therefore, in the cumulant control sense, we see that the literature on RS control actually extends backward further from the work of Jacobson in 1973. This extension is on the order of a decade. On the other hand, it seems that these observations may serve to broaden the scope of RS investigations in the future. We intend this paper as a contribution toward that effort.

In Section 2, preliminaries needed to formulate the MCV control problem are given. Then in Section 3, MCV control is defined. A Hamilton-Jacobi-Bellman (HJB) equation and associated verification theorem are derived in Section 4, and the solution structure for MCV control is found using that HJB equation in Section 5. Section 6 contains computer simulations of a representative civil structure being controlled by MCV methods, under seismic disturbances. Then in Section 7 various density and distribution graphs of the cost function for a simple MCV control example are given.

2 Preliminaries

This paper considers the Ito-sense stochastic differential equation (SDE)

$$dx(t) = f(t, x(t))dt + \sigma(t, x(t))dw(t), \quad t \in T, x(t_0) = x_0 \tag{4}$$

where $T = [t_0, t_F]$, $x(t) \in \mathbb{R}^n$ is the state, x_0 is a random variable which is independent of w, and $w(t)$ is a Brownian, not necessarily standard, motion of dimension d defined on a probability space $(\Omega, \mathcal{F}, P)$. The above SDE is interpreted in terms of the stochastic integral equation,

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))ds + \int_{t_0}^{t} \sigma(s, x(s))dw(s), \tag{5}$$

where the second integral is of the Ito type and where equality is w.p.1. In the sequel, $\mathbb{R}^n$ is equipped with an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with the usual action $\langle a, b \rangle = \sum_{i=1}^{n} a_i b_i$. This then yields a norm in the customary manner $|x|^2 = \langle x, x \rangle$. The following conditions are sufficient for existence and uniqueness of $x(t)$. See [7, page 118]. Let $Q_0 = (t_0, t_F) \times \mathbb{R}^n$, and let $\bar{Q}_0 = T \times \mathbb{R}^n$ denote the closure of Q_0. Assume that $f : \bar{Q}_0 \to \mathbb{R}^n$ and $\sigma : \bar{Q}_0 \to \mathbb{R}^n \times \mathbb{R}^d$ are Borel measurable mappings [28, page 50]. Suppose further that there exists a positive constant C such that, for all $(t, x) \in \bar{Q}_0$, $|f(t, x)| \leq C(1 + |x|)$ and $|\sigma(t, x)| \leq C(1 + |x|)$. For any bounded $B \subset \mathbb{R}^n$ and $t_0 < t_1 < t_F$ assume that there exists a constant K, which may depend upon B and t_1, such that, for all $x, y \in B$ and $t_0 \leq t \leq t_1$, $|f(t, x) - f(t, y)| \leq K|x - y|$ and $|\sigma(t, x) - \sigma(t, y)| \leq K|x - y|$. Then [7, page 118] if $E\{|x(t_0)|^2\} < \infty$, the solution of (5) exists and is unique. Furthermore, if $E\{|x(t_0)|^m\} < \infty, m = 1, 2, \cdots$, then $E|x(t)|^m$ is bounded for $m = 1, 2, \cdots$, and $t_0 \leq t \leq t_F$. Moreover, the process is a Markov diffusion process; and transition functions become available [7, page 123].

Now consider the SDE with control,

$$dx(t) = f(t, x(t), u(t))dt + \sigma(t, x(t))dw(t), \quad t \in T, x(t_0) = x_0, \tag{6}$$

when $u(t) \in U$ is the control action. Let p and q be natural numbers. Suppose that $D \subset \mathbb{R}^p$ and that $h : D \to \mathbb{R}^q$. Assume $f : \bar{Q}_0 \times U \to \mathbb{R}^n$ is $C^1(\bar{Q}_0 \times U)$, and $\sigma : \bar{Q}_0 \to \mathbb{R}^{n \times d}$ is $C^1(\bar{Q}_0)$. Furthermore assume $|f(t,0,0)| \leq c, |\sigma(t,0)| \leq c, \left|\frac{\partial f(t,x,u)}{\partial x}\right| + \left|\frac{\partial f(t,x,u)}{\partial u}\right| \leq \bar{c}$, and $\left|\frac{\partial \sigma(t,x)}{\partial x}\right| \leq \bar{c}$ for $(t,x,u) \in \bar{Q}_0 \times U, (t,x) \in \bar{Q}_0$, and constants c and $\bar{c}$.

In order to control the performance of (6), a memoryless feedback *control law* is introduced in the manner $u(t) = k(t, x(t)), t \in T$, where k is a nonrandom function with random arguments. Then (6) can be written as

$$dx(t) = f^k(t, x(t))dt + \sigma(t, x(t))dw(t), \quad t \in T, x(t_0) = x_0, \tag{7}$$

where $f^k(t,x)$ denotes $f(t,x,k(t,x))$. This is in a form similar to (4).

Now we admit only bounded, Borel measurable feedback control laws, $k(t,x) : \bar{Q}_0 \to U$ which satisfy a local Lipschitz condition. Thus, for any bounded $B \subset \mathbb{R}^n$ and $t_0 < t_1 < t_F$ there exists a constant c_1, which may depend upon B and t_1, such that, for all $x, y \in B$ and $t_0 \leq t \leq t_1, |k(t,x) - k(t,y)| \leq c_1|x - y|$. Moreover, we require that $k(t,x)$ satisfy the linear growth condition $|k(t,x)| \leq c_2(1 + |x|), \forall (t,x) \in \bar{Q}_0$, for a constant c_2. A feedback control law k which satisfies both of these conditions is called *admissible.* Then the previous existence result of (4) is applicable, and a unique solution process $x(t)$ of (7) exists [7, page 156].

Now let $\hat{P}(t,x;s,B;k)$ be the transition function for (6), which is defined as

$$\hat{P}(t,x;s,B;k) = P[x(s) \in B | x(t) = x; u(\alpha) = k(\alpha, x(\alpha)), t \leq \alpha \leq s] \quad \forall B \in \mathcal{B}(\mathbb{R}^n) \tag{8}$$

where $t_0 \leq t < s$, and $\mathcal{B}(\mathbb{R}^n)$ denotes the Borel σ-algebra, namely the least σ-algebra containing all open subsets of $\mathbb{R}^n$ [9, page 126]. Let $p(t,x,s,y;k)$ be the probability density corresponding to (8) so that $\hat{P}(t,x;s,B;k) = \int_B p(t,x,s,y;k)\, dy, \forall B \in \mathcal{B}(\mathbb{R}^n)$. This density satisfies the backward Fokker-Planck (or Kolmogorov) equation [7, 18, 33] $-\frac{\partial p(t,x;s,y;k)}{\partial t} =$

$\mathcal{O}(k)[p(t,x;s,y;k)],\quad s>t$, where $\mathcal{O}(k)$ is the backward evolution operator given by

$$\mathcal{O}(k)=\frac{\partial}{\partial t}+\left\langle f(t,x,k(t,x)),\frac{\partial}{\partial x}\right\rangle+\frac{1}{2}tr\left(\sigma(t,x)W(t)\sigma'(t,x)\frac{\partial^2}{\partial x^2}\right) \tag{9}$$

in which

$$\begin{aligned}\left\langle f(t,x,k),\frac{\partial}{\partial x}\right\rangle &= \sum_{i=1}^{n} f_i(t,x,k)\frac{\partial}{\partial x_i}\triangleq\mathcal{O}^{(1)}(k)\\ \frac{1}{2}\,\mathrm{tr}\,\left(\sigma(t,x)W(t)\sigma'(t,x)\frac{\partial^2}{\partial x^2}\right) &= \frac{1}{2}\sum_{i,j=1}^{n}(\sigma(t,x)W(t)\sigma'(t,x))_{ij}\frac{\partial^2}{\partial x_i\partial x_j}\triangleq\mathcal{O}^{(2)}.\end{aligned} \tag{10}$$

In (9), tr denotes the trace operator. The derivative operators are defined so that the i-th element in the n-tuple $(\partial/\partial x)$ is $(\partial/\partial x_i)$, which we shall choose to regard as column vectors, and the ij-th element in the $n\times n$ matrix operator $(\partial^2/\partial x^2)$ is $(\partial^2/\partial x_i\partial x_j)$. The matrix $W(t)$ characterizes $w(t)$ in the manner $E\{dw(t)dw'(t)\}=W(t)dt$ where the independent increments $dw(t)$ are assumed to be zero-mean Gaussian random variables, and the superscript $(')$ denotes transposition.

For all $(t,x)\in\bar{Q}_0$, a real valued function $\Phi(t,x)$ on $T\times\mathbb{R}^n$ satisfies a polynomial growth condition, if there exist constants k_1 and k_2 such that $|\Phi(t,x)|\leq k_1(1+|x|^{k_2})$. Let $C^{1,2}(\bar{Q}_0)$ denote the space of $\Phi(t,x)$ such that Φ and the partial derivatives $\Phi_t,\Phi_{x_i},\Phi_{x_ix_j}$ for $i,j=1,\cdots,n$ are continuous on $\bar{Q}_0$. Also let $C_p^{1,2}(\bar{Q}_0)$ denote the space of $\Phi(t,x)\in C^{1,2}(\bar{Q}_0)$ such that $\Phi,\Phi_t,\Phi_{x_i},\Phi_{x_ix_j}$ for $i,j=1,\cdots,n$ satisfy a polynomial growth condition. Assumptions $\Phi(t,x)\in C_p^{1,2}(\bar{Q}_0)$, k admissible, and $E\{|x(s)|^m|x(t)=x\}$ bounded for $m=1,2,\cdots$ and $t\leq s\leq t_F$ ensure existence of the terms in the right member of the Dynkin formula (see [9, pages 128,135,161]),

$$\Phi(t,x)=E\left\{\int_t^{t_F}-\mathcal{O}(k)\Phi(s,x(s))\,ds+\Phi(t_F,x(t_F))\middle|\,x(t)=x\right\}. \tag{11}$$

In the sequel, we shorten this expectation notation to E_{tx}. In order to assess the performance of (6), consider the cost function (1)

$$J(t,x(t),k) = \int_t^{t_F} \Big[L(s,x(s),k(s,x(s))) \Big] ds + \psi(x(t_F)). \tag{12}$$

Assume that L and ψ satisfy the polynomial growth conditions $|L(t,x,u)| \leq c_3\ (1+|x|+|u|)^{c_4}$, $\forall (t,x,u) \in \bar{Q}_0 \times U, |\psi(x)| \leq c_3\ (1+|x|)^{c_4}$, $\forall x \in \mathbb{R}^n$, for constants c_3 and c_4. Fleming and Rishel show that a process $x(t)$ from (6), having an admissible controller k, with the assumptions above, is such that $E_{tx}\{J(t,x(t),k)\}$ is finite [7, page 157]. Lastly, for real symmetric matrices A and B, we denote $A \geq B$ if $A-B$ is positive semidefinite, and $A > B$ if $A-B$ is positive definite.

3 Minimal Cost Variance Control Problem

This section deals with the definition of the minimal cost variance (MCV) control problem in the completely observed, or full-state-feedback, case. MCV control is a type of cumulant control where we minimize the variance of the cost function while keeping the mean of the cost function at a specified level.

The class of admissible control laws, and comparison of control laws within the class, is defined in terms of the first and second moments of the cost. Introduce the notation $V_1(t,x;k) = E_{tx}\{J(t,x(t),k)\}$ and $V_2(t,x;k) = E_{tx}\{J^2(t,x(t),k)\}$.

Definition 1: A function $M : \bar{Q}_0 \to \mathbb{R}^+$, which is $C^{1,2}(\bar{Q}_0)$, is an *admissible mean cost function* if there exists an admissible control law k such that $V_1(t,x;k) = M(t,x)$ for $t \in T$ and $x \in \mathbb{R}^n$.

Remark: One can of course think of generating a plethora of admissible mean cost functions. All that is necessary, in principle, is to choose a stabilizing admissible control law and then to evaluate the mean cost. In practice, there is the task of representing this mean cost, on the one hand, the task of computing it on the other, and the background question of existence. In the present paper, the approach taken is to solve the mean-cost constraint equation for all possible solutions, and then to use the remaining design freedom to achieve a minimization of the cost variance. For the explicit situations discussed in the sequel, this leads to a simultaneous solution for both admissible mean costs and cost variances. This, of course, is in the general spirit of multiplier methods; and a type of multiplier function does appear in our later applications.

Definition 2: Every admissible M defines a class K_M of *control laws* k *corresponding to* M in the manner that $k \in K_M$ if and only if k is an admissible control law which satisfies Definition 1.

It is now possible to define an MCV control law $k^*_{V|M}$.

Definition 3: Let M be an admissible mean cost function, and let K_M be its induced class of admissible control laws. An MCV control law $k^*_{V|M}$ satisfies $V_2(t,x;k^*_{V|M}) = V_2^*(t,x) \leq V_2(t,x;k)$, for $t \in T$, $x \in \mathbb{R}^n$, whenever $k \in K_M$. The corresponding minimal cost variance is given by $V^*(t,x) = V_2^*(t,x) - M^2(t,x)$ for $t \in T$, $x \in \mathbb{R}^n$.

An MCV control problem, therefore, is quite general in its scope. It presupposes that a cost mean M, not necessarily minimal, has been specified; and it seeks a control law which minimizes the variance of the cost, about M.

4 Hamilton-Jacobi-Bellman Equation for V^*

To solve the MCV control problem, we derive a Hamilton-Jacobi-Bellman (HJB) equation, under the assumption that a sufficiently smooth solution exists. A full-state-feedback MCV control law is constructed in the following section, using this HJB equation, for a large class of problems. We first present a number of preliminary results.

Lemma 4.1 *Assume that* $M \in C_p^{1,2}(\bar{Q}_0)$ *is an admissible mean cost function, and let* $k \in$

K_M be a control law corresponding to M. Then $\mathcal{O}(k)\,[M(t,x)]+L(t,x,k(t,x)) = 0$ for $t \in T$, $x \in \mathbb{R}^n$, where $M(t_F,x) = \psi(x)$.

Proof: See [21] or [33, page 117]. □

Consider an open set $Q \subset Q_0$.

Lemma 4.2 *(Verification Lemma). Let $M \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be a solution to the partial differential equation*

$$\mathcal{O}(k)[M(t,x)] + L(t,x,k(t,x)) = 0, \quad \forall (t,x) \in Q, \tag{13}$$

with the boundary condition $M(t_F,x) = \psi(x)$. Then $M(t,x) = V_1(t,x;k)$ for every $k \in K_M$ and $(t,x) \in Q$.

Proof: Because x is a Markov diffusion process, and because $M \in C_p^{1,2}(Q)$, we may use the Dynkin formula (11). From the boundary condition and the Dynkin formula we obtain

$$M(t,x) = E_{tx}\left\{\int_t^{t_F} -\mathcal{O}(k)[M(s,x(s))]\;ds + \psi(x(t_F))\right\}. \tag{14}$$

Substitute from (13) for the expression $-\mathcal{O}(k)[M(s,x(s))]$ in (14). Then we obtain

$$M(t,x) = E_{tx}\left\{\int_t^{t_F}\Big[L(s,x(s),k(s,x(s)))\Big]ds + \psi(x(t_F))\right\} = V_1(t,x;k), \tag{15}$$

for each $k \in K_M$ and $(t,x) \in Q$. □

Lemma 4.3 *Assume $L : \bar{Q}_0 \times U \to \mathbb{R}^+$ and $L \in C(\bar{Q}_0 \times U)$, together with the linear growth condition on k, and the polynomial growth condition on L and ψ. Let $x^*(t)$ be the solution of (6) when $k = k^*_{V|M}$ Then*

$$\int_t^{t+\Delta t} L(s,x(s),k(s,x(s)))ds\left[\int_{t+\Delta t}^{t_F} L(s,x^*(s),k^*_{V|M}(s,x^*(s)))ds + \psi(x^*(t_F))\right]$$

is uniformly integrable.

Proof: By Doob [6, page 629], if (we use L_s and L^* to abbreviate the longer expressions above)

$$E_{tx}\left|\int_t^{t+\Delta t} L_s ds\left[\int_{t+\Delta t}^{t_F} L^* ds+\psi(x^*(t_F))\right]\right|^\alpha$$

is bounded in t for some $\alpha > 1$, then the lemma is proved. We have

$$\int_t^{t+\Delta t} L_s ds\left[\int_{t+\Delta t}^{t_F} L^* ds+\psi(x^*(t_F))\right]=\Delta t L(t^+,x^+,k^+)[(t_F-t-\Delta t)L(t_1,x_1,k_1)+\psi(x_{t_F})]$$

where $t+\Delta t \le t_1 \le t_F, x_1 = x^*(t_1,\omega)$, $x_{t_F} = x^*(t_F,\omega)$, $k_1 = k^*(t_1,x_1)$, and $t \le t^+ \le t+\Delta t$. Consequently we obtain

$$\begin{aligned}&|\Delta t L(t^+,x^+,k^+)[(t_F-t-\Delta t)L(t_1,x_1,k_1)+\psi(x_{t_F})]| \le \\ &\quad |\Delta t||L(t^+,x^+,k^+)|\ \Big[|t_F-t-\Delta t|\ |L(t_1,x_1,k_1)|+|\psi(x_{t_F})|\Big].\end{aligned}$$

Now we use the polynomial growth conditions on L and ψ, to obtain

$$\begin{aligned}&|\Delta t L(t^+,x^+,k^+)[(t_F-t-\Delta t)L(t_1,x_1,k_1)+\psi(x_{t_F})]| \le \\ &\quad |\Delta t|\ c_3(1+|x^+|+|k^+|)^{c_4}\Big[c_3|t_F-t-\Delta t|\ (1+|x_1|+|k_1|)^{c_4}+c_3(1+|x_{t_F}|)^{c_4}\Big].\end{aligned}$$

From the linear growth condition on k,

$$|\Delta t L(t^+,x^+,k^+)[(t_F-t-\Delta t)L(t_1,x_1,k_1)+\psi(x_{t_F})]| \le |\Delta t(t_F-t-\Delta t)|\ c_5\ (1+\|x\|)^{2c_4}$$

where $\|x\|$ is the sup norm on the process segment which is a concatenation of the two subsegments which interface at $t+\Delta t$. Now we have

$$|\Delta t L(t^+,x^+,k^+)[(t_F-t-\Delta t)L(t_1,x_1,k_1)+\psi(x_{t_F})]| \le c_6(1+\|x\|)^{2c_4},$$

and thus

$$E_{tx}|\Delta t L(t^+,x^+,k^+)[(t_F-t-\Delta t)L(t_1,x_1,k_1)+\psi(x_{t_F})]|^\alpha \le E_{tx}[c_6(1+\|x\|)]^{2c_4\alpha}.$$

It can be shown [9, Appendix D] that $E_{tx}\|x\|^m < \infty$ for $m = 1, 2, \cdots$. Take $\alpha = \frac{m}{2c_2} > 1$. Then $E_{tx}|\Delta t L(t^+, x^+, k^+)[(t_F - t - \Delta t)L(t_1, x_1, k_1) + \psi(x_{t_F})]|^\alpha$ is bounded, and by Doob [6, page 629] we have uniform integrability, as desired. □

Next we derive the HJB equation for the second moment of the cost function, in the following theorem, which, as we observed early in the section, assumes the existence of an optimal controller. Then we present the verification theorem which is a sufficient condition for constructing a minimum.

Theorem 4.1 *Let M be an admissible mean cost function and K_M be the corresponding class of control laws. Assume the existence of an optimal controller $k^*_{V|M}$ and an optimum value function $V_2^* \in C_p^{1,2}(\bar{Q}_0)$. Then $k^*_{V|M}$ and V_2^* satisfy the partial differential equation*

$$\mathcal{O}(k^*_{V|M})[V_2^*(t,x)] + 2M(t,x)L(t,x,k^*_{V|M}(t,x)) = 0 \tag{16}$$

for $t \in T$, $x \in \mathbb{R}^n$, where

$$\begin{aligned} &\mathcal{O}(k^*_{V|M})[V_2^*(t,x)] + 2M(t,x)L(t,x,k^*_{V|M}(t,x)) \\ = &\min_{k \in K_M} \left\{ \mathcal{O}(k)[V_2^*(t,x)] + 2M(t,x)L(t,x,k(t,x)) \right\}, \end{aligned} \tag{17}$$

along with the boundary condition $V_2^*(t_F, x) = M^2(t_F, x) = \psi^2(x)$, $x \in \mathbb{R}^n$.

Proof: Define a controller $k_1 \in K_M$ by the action,

$$k_1(r,x) = \begin{cases} k(r,x), & t \le r \le t + \Delta t \\ k^*_{V|M}(r,x), & t + \Delta t < r \le t_F; \end{cases} \tag{18}$$

then the second moment is given by

$$\begin{aligned} V_2(t,x;k_1) &= E_{tx}\{J^2(t,x(t),k_1)\} \\ &= E_{tx}\left\{ \left[\int_t^{t+\Delta t} L_s ds + \int_{t+\Delta t}^{t_F} L^* ds + \psi(x^*(t_F)) \right]^2 \right\} \end{aligned}$$

$$= E_{tx}\left\{\left[\int_t^{t+\Delta t} L_s ds\right]^2\right\} + 2E_{tx}\left\{\left[\int_t^{t+\Delta t} L_s ds\right]\left[\int_{t+\Delta t}^{t_F} L^* ds + \psi(x^*(t_F))\right]\right\} + E_{tx}\left\{\left[\int_{t+\Delta t}^{t_F} L^* ds + \psi(x^*(t_F))\right]^2\right\}$$

where $L^* = L(s, x^*(s), k^*_{V|M}(s, x^*(s)))$, $x^*(t)$ is the solution of (6) when $k = k^*_{V|M}$ and $L_s = L(s, x(s), k(s, x(s)))$. By definition $V_2^*(t,x) \leq V_2(t,x;k_1)$. Now we can substitute for $V_2(t,x;k_1)$ and obtain

$$\begin{aligned} V_2^*(t,x) \leq\ & E_{tx}\left\{\left[\int_t^{t+\Delta t} L_s ds\right]^2\right\} + 2E_{tx}\left\{\left[\int_t^{t+\Delta t} L_s ds\right]\left[\int_{t+\Delta t}^{t_F} L^* ds + \psi(x^*(t_F))\right]\right\} \\ & + E_{tx}\left\{\left[\int_{t+\Delta t}^{t_F} L^* ds + \psi(x^*(t_F))\right]^2\right\}. \end{aligned}$$

We can apply the mean value theorem for each sample function. Then

$$\begin{aligned} V_2^*(t,x) \leq\ & \Delta t^2 E_{tx}\{L^2(t^+, x^+, k^+)\} + 2\Delta t E_{tx}\Big\{L(t^+, x^+, k^+) \\ & \left[\int_{t+\Delta t}^{t_F} L^* ds + \psi(x^*(t_F))\right]\Big\} + E_{tx}\{V_2^*(t+\Delta t, x(t+\Delta t))\}, \end{aligned}$$

where the last term in the right side uses Chapman-Kolmogorov equation. Because of the assumptions we have made, we can use the Dynkin formula (11),

$$E_{tx}\{V_2^*(t+\Delta t, x(t+\Delta t))\} - V_2^*(t,x) = E_{tx}\left\{\int_t^{t+\Delta t} \mathcal{O}(k)[V_2^*(r, x(r))]dr\right\}.$$

Thus, we obtain

$$\begin{aligned} V_2^*(t,x) \leq\ & (\Delta t)^2 E_{tx}\{L^2(t^+, x^+, k^+)\} + 2\Delta t E_{tx}\{L(t^+, x^+, k^+) \\ & \left[\int_{t+\Delta t}^{t_F} L^* ds + \psi(x^*(t_F))\right]\} + E_{tx}\left\{\int_t^{t+\Delta t} \mathcal{O}(k)[V_2^*(r, x(r))]dr\right\} \\ & + V_2^*(t,x). \end{aligned} \quad (19)$$

We also have [9, page 164]

$$\lim_{\Delta t \to 0} E_{tx}\left\{L(t^+, x^+, k^+)\left[\int_{t+\Delta t}^{t_F} L^* ds + \psi(x^*(t_F))\right]\right\} = M(t,x)L(t,x,k(t,x))$$

because of the uniform integrability condition given in Lemma 4.3, $k^*_{V|M} \in K_M$, and the fact that, as Δt goes to zero, $L(t^+, x^+, k^+)$ approaches $L(t, x, k(t,x))$. The third term in the right member of (19) can be treated in a similar manner, but we omit details for reasons of brevity. Now divide (19) by Δt and let $\Delta t \to 0$ to get $0 \le \mathcal{O}(k)[V_2^*(t,x)] + 2M(t,x)L(t,x,k(t,x))$. Equality holds if $k(t,x) = k^*_{V|M}(t,x)$ where $k^*_{V|M}$ is an optimal feedback control law. This concludes the proof. See also [22]. □

The verification theorem for the second moment case is now presented. This verification theorem states that if there exists a sufficiently smooth solution of the HJB equation, then it is the optimal cost of control, and using this solution an optimal feedback control law can be determined.

Theorem 4.2 *(Verification Theorem). Let $V_2^* \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be a nonnegative solution to the partial differential equation*

$$0 = \min_{k \in K_M} \{2M(t,x)L(t,x,k(t,x)) + \mathcal{O}(k)[V_2^*(t,x)]\}, \ \forall (t,x) \in Q \tag{20}$$

with the boundary condition $V_2^*(t_F, x) = \psi^2(x)$. Then $V_2^*(t,x) \le V_2(t,x;k)$ for every $k \in K_M$ and any $(t,x) \in Q$. If in addition such a k also satisfies the equation

$$\begin{gathered} 2M(t,x)L(t,x,k(t,x)) + \mathcal{O}(k)[V_2^*(t,x)] = \\ \min_{\tilde{k} \in K_M} \left\{2M(t,x)L(t,x,\tilde{k}(t,x)) + \mathcal{O}(\tilde{k})[V_2^*(t,x)]\right\} \end{gathered}$$

for all $(t,x) \in Q$, then $V_2^*(t,x) = V_2(t,x;k)$ and $k = k^*_{V|M}$ is an optimal control law.

Proof: From (20) for each $k \in K_M$ and $(t,x) \in Q$

$$2M(t,x)L(t,x,k(t,x)) + \mathcal{O}(k)[V_2^*(t,x)] \ge 0. \tag{21}$$

Because x is a Markov diffusion process, and because $V_2^* \in C_p^{1,2}(Q)$, we may use the Dynkin formula (11). From the boundary condition and the Dynkin formula we obtain

$$V_2^*(t,x) = E_{tx}\left\{\int_t^{t_F} -\mathcal{O}(k)[V_2^*(s,x(s))]\, ds + \psi^2(x(t_F))\right\}. \tag{22}$$

Let $L_s = L(s, x(s), k(s, x(s)))$ and $L_r = L(r, x(r), k(r, x(r)))$, and substitute from (21) expression $-\mathcal{O}(k)[V_2^*(s, x(s))]$ in (22). Then we obtain

$$\begin{aligned}
V_2^*(t,x) &\leq E_{tx}\left\{\int_t^{t_F} 2L_s M(s,x(s))\, ds + \psi^2(x(t_F))\right\} \\
&= E_{tx}\left\{\int_t^{t_F} 2L_s E_{sx}\left\{\int_s^{t_F} L_r\, dr + \psi(x(t_F))\right\}\, ds + \psi^2(x(t_F))\right\} \\
&= E_{tx}\left\{\int_t^{t_F} E_{sx}\left\{2L_s \int_s^{t_F} L_r dr + 2L_s\psi(x(t_F))\right\} ds\right\} + E_{tx}\left\{\psi^2(x(t_F))\right\} \\
&= \int_t^{t_F} E_{tx}\left\{E_{sx}\left\{2L_s \int_s^{t_F} L_r dr + 2L_s\psi(x(t_F))\right\}\right\} ds + E_{tx}\left\{\psi^2(x(t_F))\right\}.
\end{aligned}$$

For a justification of interchange of the integral and the expectation in the last equation, see [6, page 62] and [5, page 65]. Consequently, we have

$$\begin{aligned}
V_2^*(t,x) &\leq \int_t^{t_F} E_{tx}\left\{\left[2L_s \int_s^{t_F} L_r dr + 2L_s\psi(x(t_F))\right]\right\} ds + E_{tx}\left\{\psi^2(x(t_F))\right\} \\
&= E_{tx}\left\{\int_t^{t_F} \left[2L_s \int_s^{t_F} L_r dr + 2L_s\psi(x(t_F))\right]\, ds + \psi^2(x(t_F))\right\} \\
&= E_{tx}\left\{\int_t^{t_F} L_s ds \int_t^{t_F} L_r dr + 2\int_t^{t_F} L_s\psi(x(t_F))\, ds + \psi^2(x(t_F))\right\} \\
&= E_{tx}\left\{\left[\int_t^{t_F} L_s ds + \psi(x(t_F))\right]^2\right\} = V_2(t,x;k).
\end{aligned}$$

This proves the first part. For the second part, the inequality becomes equality. □

Note that if M is given a priori, then the optimal second moment results in the optimal variance.

With these results in hand, it is then possible to transfer the results to the cumulant functions by means of the following pair of theorems, which make use of the notation $|a|_A^2 = a'Aa$.

Theorem 4.3 *Let $M \in C_p^{1,2}(\bar{Q}_0)$ be an admissible mean cost function, and let M induce a non-empty class K_M of admissible control laws. Assume the existence of an optimal control law $k = k^*_{V|M}$ and an optimum value function $V^* \in C_p^{1,2}(\bar{Q}_0)$. Then the MCV function V^**

satisfies the HJB equation

$$\min_{k\in K_M} \mathcal{O}(k)[V^*(t,x)] + \left|\frac{\partial M(t,x)}{\partial x}\right|^2_{\sigma(t,x)W(t)\sigma'(t,x)} = 0, \tag{23}$$

for $(t,x) \in \bar{Q}_0$*, together with the terminal condition,* $V^*(t_F, x) = 0$*.*

Proof: From Definition 3 and (20), it follows that

$$\min_{k\in K_M} \left\{\mathcal{O}(k)[V^*(t,x) + M^2(t,x)] + 2M(t,x)L(t,x,k(t,x))\right\} = 0.$$

To prove (23), it is necessary and sufficient to establish that

$$\mathcal{O}(k)[M^2(t,x)] + 2M(t,x)L(t,x,k(t,x)) = \left|\frac{\partial M(t,x)}{\partial x}\right|^2_{\sigma(t,x)W(t)\sigma'(t,x)} \tag{24}$$

whenever $k \in K_M$. In order to see this, write $\mathcal{O}(k)$ as the sum $\mathcal{O}^{(1)}(k) + \mathcal{O}^{(2)}$ of two operators, where $\mathcal{O}^{(i)}$ involves the partial derivative of exactly the i-th order; see (10). Then $\mathcal{O}^{(1)}(k)[M^2(t,x)] = 2M(t,x)\mathcal{O}^{(1)}(k)[M(t,x)]$, which by Lemma 4.1 becomes

$$\mathcal{O}^{(1)}(k)[M^2(t,x)] = -2M(t,x)\left[\mathcal{O}^{(2)}[M(t,x)] + L(t,x,k(t,x))\right]. \tag{25}$$

Substitute in (24)) to get

$$\mathcal{O}^{(2)}[M^2(t,x)] - 2M(t,x)\mathcal{O}^{(2)}[M(t,x)] = \left|\frac{\partial M(t,x)}{\partial x}\right|^2_{\sigma(t,x)W(t)\sigma'(t,x)} \tag{26}$$

for $t \in T$, $x \in \mathbb{R}^n$. Note that k does not appear explicitly in equation (26). Note also that because $\mathcal{O}^{(2)} = \frac{1}{2}tr\left(\sigma(t,x)W(t)\sigma'(t,x)\frac{\partial^2}{\partial x^2}\right)$, (26) is equivalent to

$$\frac{1}{2}tr\left(\sigma(t,x)W(t)\sigma'(t,x)\left[\frac{\partial^2(M^2(t,x))}{\partial x^2} - 2M(t,x)\frac{\partial^2 M(t,x)}{\partial x^2}\right]\right) = \left|\frac{\partial M(t,x)}{\partial x}\right|^2_{\sigma(t,x)W(t)\sigma'(t,x)}. \tag{27}$$

However,

$$\frac{\partial^2 M^2(t,x)}{\partial x^2} - 2M(t,x)\frac{\partial^2 M(t,x)}{\partial x^2} = 2\frac{\partial M(t,x)}{\partial x}\left(\frac{\partial M(t,x)}{\partial x}\right)',$$

which combines with (27) to establish (24). □

Theorem 4.4 *(Verification Theorem). Let M be an admissible mean cost function satisfying $M^2(t,x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$, and let K_M be the associated non-empty class of admissible control laws. Suppose that a nonnegative function $V^* \in C_p^{1,2}(Q) \cap C(\bar{Q})$ is a solution to the partial differential equation*

$$\min_{k \in K_M} \mathcal{O}(k)[V^*(t,x)] + \left| \frac{\partial M(t,x)}{\partial x} \right|^2_{\sigma(t,x)W(t)\sigma'(t,x)} = 0, \quad \forall (t,x) \in Q, \tag{28}$$

together with the boundary condition $V^*(t_F,x) = 0$. Then $V^*(t,x) \leq V(t,x;k)$ for every $k \in K_M$ and any $(t,x) \in Q$. If in addition such a k satisfies the equation

$$\mathcal{O}(k)[V^*(t,x)] = \min_{\tilde{k} \in K_M} \left\{ \mathcal{O}(\tilde{k})[V^*(t,x)] \right\}$$

for all $(t,x) \in Q$, then $V^*(t,x) = V(t,x;k)$ and $k = k^*_{V|M}$ is an optimal control law.

Proof: From (28) for each $k \in K_M$ and $(t,x) \in Q$

$$\mathcal{O}(k)[V^*(t,x)] + \left| \frac{\partial M(t,x)}{\partial x} \right|^2_{\sigma(t,x)W(t)\sigma'(t,x)} \geq 0. \tag{29}$$

Because x is a Markov diffusion process, and because $V^*, M^2 \in C_p^{1,2}(Q)$, we may use the Dynkin formula (11) to obtain

$$V^*(t,x) = E_{tx} \left\{ \int_t^{t_F} -\mathcal{O}(k)[V^*(s,x(s))] \, ds \right\} \tag{30}$$

and

$$M^2(t,x) = E_{tx} \left\{ \int_t^{t_F} -\mathcal{O}(k)[M^2(s,x(s))] \, ds + \psi^2(x(t_F)) \right\}. \tag{31}$$

By (29) and (30), we get, with the aid of the notation $L_s = L(s,x(s),k(s,x(s)))$,

$$\begin{aligned} V^*(t,x) &\leq E_{tx} \left\{ \int_t^{t_F} \left| \frac{\partial M(s,x(s))}{\partial x} \right|^2_{\sigma(s,x(s))W(s)\sigma'(s,x(s))} ds \right\} \\ &= E_{tx} \left\{ \int_t^{t_F} \left[\mathcal{O}(k)[M^2(s,x(s))] + 2M(s,x(s))L_s \right] \, ds \right\}, \end{aligned}$$

where we used (24) to obtain the last expression. Use Definition 3 and (31) to get

$$V_2^*(t,x) \leq E_{tx} \left\{ \int_t^{t_F} 2M(s,x(s))L_s \, ds + \psi^2(x(t_F)) \right\}.$$

By the analysis in the proof of Theorem 4.2, the above inequality becomes $V_2^*(t,x) \leq V_2(t,x;k)$, which in turn implies $V^*(t,x) \leq V(t,x;k)$. $\square$

Equation (28) in Theorem 4.4 differs from the classical HJB result for minimal mean controllers in that the integrand function L does not appear explicitly. In order to compare $k^*_{V|M}$ with k^*_M, recall that V_1^* satisfies (see for example [33])

$$\min_k \Big[\mathcal{O}(k)[V_1^*(t,x)] + L(t,x,k(t,x)) \Big] = 0 \tag{32}$$

together with the terminal condition $V_1^*(t_F,x) = \psi(x)$. Equation (23) may therefore be compared with a minimal mean problem in which

$$L(t,x,k(t,x)) = \left| \frac{\partial M(t,x)}{\partial x} \right|^2_{\sigma(t,x)W(t)\sigma'(t,x)}, \tag{33}$$

provided that it is realized in (23) that the control law is constrained to be in the class K_M, whereas in (32) it is constrained only to be an admissible function of its arguments. Thus an analogy between the MCV problem and a minimal mean problem with control law constraints, but no cost of control action, can be drawn immediately. The precise nature of this analogy depends upon the nature of f.

5 Solutions of MCV Control

In this section we derive the full-state-feedback solution of the MCV control problem for a linear system and a quadratic cost function. Here we will look for an admissible linear controller that minimizes the variance of the cost function. We consider the class of admissible controls that satisfy the equation $L(t,x,k(t,x)) + \mathcal{O}(k)[M(t,x)] = 0$. We assume that L, f, and M are given, and we wish to find k. Henceforward we will write $\sigma(t,x) = E(t)$, and make the assumptions

$$\sigma(t,x) = E(t), \tag{34}$$

$$L(t, x, k(t, x)) = h(t, x) + k'(t, x)R(t)k(t, x), \tag{35}$$

$$\psi(x(t_F)) = x'(t_F)Q_F x(t_F), \tag{36}$$

$$f(t, x, k(t, x)) = g(t, x) + B(t)k(t, x), \tag{37}$$

where k is an admissible feedback control law; $h : \bar{Q}_0 \to \mathbb{R}^+$ is $C(\bar{Q}_0)$ and satisfies the polynomial growth conditions assumed for L; and $g : \bar{Q}_0 \to \mathbb{R}^n$ is $C^1(\bar{Q}_0)$ and satisfies the linear growth condition and the local Lipschitz condition assumed for f. Moreover $E(t), R(t) > 0$, and $B(t)$ are continuous real matrices of appropriate dimensions for all $t \in T$.

Here we state the results of Liu and Leake [17]. Let $x \in \mathbb{R}^n$ be a real n-vector, $z(x)$ and $y(x)$ be real r-vector functions, and $\alpha(x)$ be a real function defined on $\mathbb{R}^n$.

Lemma 5.1 *Let $y(x)$ and $\alpha(x)$ be given. Then there exists $z(x)$ which satisfies the condition*

$$\langle z(x), z(x)\rangle + 2\langle z(x), y(x)\rangle + \alpha(x) = 0 \tag{38}$$

if and only if $|y(x)|^2 \geq \alpha(x)$. In such a case, the set of all solutions to (38) is represented by

$$z(x) = \beta(x)a(x) - y(x) \tag{39}$$

where $\beta(x) = (|y(x)|^2 - \alpha(x))^{\frac{1}{2}}$ and $a(x)$ is an arbitrary unit vector.

Proof: The sufficiency follows by direct evaluation. To show the conditions are necessary, note that $|y|^2 < \alpha$ implies that $|z + y|^2 < 0$, which is a contradiction. Let $w = z + y$, then (38) implies that $\langle w, w\rangle = \beta^2$; taking $a = \dfrac{w}{|w|}$, we have $w = \beta a$, and (39) follows. □

Lemma 5.2 *(Liu and Leake Lemma). Let X be a positive definite symmetric-real matrix. Then there exists $z(x)$ which satisfies the condition*

$$\langle z(x), Xz(x)\rangle + 2\langle z(x), y(x)\rangle + \alpha(x) = 0 \tag{40}$$

if and only if $\langle y(x), X^{-1}y(x)\rangle \geq \alpha(x)$. *In this case, the set of all solutions to (40) is represented by*

$$z(x) = \beta(x)H^{-1}a(x) - X^{-1}y(x) \tag{41}$$

where

$$\beta(x) = \Big(\langle y(x), X^{-1}y(x)\rangle - \alpha(x)\Big)^{\frac{1}{2}}, \tag{42}$$

H *is a non-singular matrix such that* $X = H'H$, *and* $a(x)$ *is an arbitrary unit vector.*

Proof: The existence of such an H is well known. One instance is sometimes denoted $X^{\frac{1}{2}}$. No difficulty accrues due to the non-uniqueness of H, because it is subsumed into the unit vector $a(x)$. The proof of this lemma follows from Lemma 5.1 by a change of variables $\hat{z} = Hz$ and $\hat{y} = (H^{-1})'y$. □

Remark: The presence of the term involving the unit vector $a(x)$ in (39) is worthy of reflection. The only way in which this term can vanish is for $\beta(x)$ to vanish. Later, in the application of this result, the term will appear in a product with another vector Gx, for G a matrix, and will be chosen to be in the direction opposite to that of Gx. This means that, whenever x approaches the null space of G, it is not possible to define a unique limiting value for the unit vector. Therefore, in order for the term to remain continuous as x approaches the null space of G, we shall simply require that $\beta(x)$ then approach zero.

Lemma 5.3 *Assume that* $M \in C_p^{1,2}(Q) \cap C(\bar{Q})$ *and that the above assumptions (34)–(37) are satisfied. Then we have a solution* $k(t,x)$*, which may or may not be admissible, if and only if*

$$\begin{aligned}&\frac{1}{4}\left(\frac{\partial M(t,x)}{\partial x}\right)' B(t)R^{-1}(t)B'(t)\left(\frac{\partial M(t,x)}{\partial x}\right)\\ &\quad\geq \frac{\partial M(t,x)}{\partial t} + \frac{1}{2}\ tr\ \left(E(t)W(t)E'(t)\frac{\partial^2 M(t,x)}{\partial x^2}\right) + h(t,x) + g'(t,x)\left(\frac{\partial M(t,x)}{\partial x}\right).\end{aligned}$$

Proof: Rewriting,

$$\frac{\partial M(t,x)}{\partial t} + L(t,x,k(t,x)) \quad + \quad \frac{1}{2} tr\left(\sigma(t,x)W(t)\sigma'(t,x)\frac{\partial^2 M(t,x)}{\partial x^2}\right)$$
$$+f'(t,x,k(t,x))\frac{\partial M(t,x)}{\partial x} = 0. \tag{43}$$

By substituting expressions for (34), (35) and (37) into (43) and suppressing the arguments, we obtain,

$$\frac{\partial M}{\partial t} + h + k'Rk + \frac{1}{2} tr\left(EWE'\frac{\partial^2 M}{\partial x^2}\right) + g'\left(\frac{\partial M}{\partial x}\right) + k'B'\left(\frac{\partial M}{\partial x}\right) = 0. \tag{44}$$

One can then solve the above equation for k, using the method of Liu and Leake; see Lemma 5.2. We may identify from (44) $z \Leftrightarrow k, X \Leftrightarrow R, y \Leftrightarrow \frac{1}{2}B'\frac{\partial M}{\partial x}$, and $\alpha \Leftrightarrow \frac{\partial M}{\partial t} + \frac{1}{2}tr\left(EWE'\frac{\partial^2 M}{\partial x^2}\right) + h + g'\frac{\partial M}{\partial x}$. Accordingly, we must satisfy

$$\frac{1}{4}\left(\frac{\partial M}{\partial x}\right)' BR^{-1}B'\left(\frac{\partial M}{\partial x}\right) \geq \frac{\partial M}{\partial t} + \frac{1}{2} tr\left(EWE'\frac{\partial^2 M}{\partial x^2}\right) + h + g'\left(\frac{\partial M}{\partial x}\right), \tag{45}$$

which corresponds to $\langle y(x), X^{-1}y(x)\rangle \geq \alpha(x)$ condition in Lemma 5.2. □

Now we are ready to characterize all controllers, $k \in K_M$.

Theorem 5.1 *Assume that the condition of Lemma 5.3 is satisfied. Then a control law k is in K_M if and only if (1) it is admissible and (2) it is of the form*

$$k(t,x) = \beta(x)H^{-1}a(x) - \frac{1}{2}R^{-1}(t)B'(t)\left(\frac{\partial M(t,x)}{\partial x}\right), \tag{46}$$

where $a(x)$ is an arbitrary unit vector, $H'H = R$, and

$$\beta(x) = \sqrt{\frac{1}{4}\left(\frac{\partial M}{\partial x}\right)' BR^{-1}B'\left(\frac{\partial M}{\partial x}\right) - \frac{\partial M}{\partial t} - \frac{1}{2}\ tr\ \left(EWE'\frac{\partial^2 M}{\partial x^2}\right) - h - g'\left(\frac{\partial M}{\partial x}\right)}. \tag{47}$$

Moreover, $\beta(x) = 0$ corresponds to the optimal mean cost law.

Proof: See Lemmas 5.2 and 5.3. Now for any admissible control law, we have from (44) the equation

$$\left[\frac{\partial M}{\partial t} + \frac{1}{2}tr\left(EWE'\frac{\partial^2 M}{\partial x^2}\right) + h + g'\frac{\partial M}{\partial x}\right] + k'Rk + k'B'\frac{\partial M}{\partial x} = 0. \tag{48}$$

Incorporating (45), we obtain

$$\left(k'Rk + k'B'\left(\frac{\partial M}{\partial x}\right)\right) \geq -\frac{1}{4}\left(\frac{\partial M}{\partial x}\right)' BR^{-1}B'\left(\frac{\partial M}{\partial x}\right). \tag{49}$$

The left hand side of the above inequality is $-\alpha$ (see (48)), and the right hand side is $-\langle y, X^{-1}y\rangle$. Thus if k is the minimal mean cost law then $\beta = 0$ and $M(t,x) = V_1(t,x;k) = V_1^*(t,x)$. If k is not the minimal mean cost law, and (45) is satisfied, then $\beta > 0$. □

For a wide variety of problems, therefore, we have shown that sub-optimal mean control introduces the possibility of reducing variance, that is, we have some freedom in selecting $a(x)$.

To find the solution of the MCV control problem, we rewrite the HJB equation (28) of Theorem 4.4 as

$$\begin{aligned}\frac{-\partial V^*(t,x)}{\partial t} &= \min_{k\in K_M}\left[f'(t,x,k(t,x))\frac{\partial V^*(t,x)}{\partial x}\right. \\ &\quad +\frac{1}{2}tr\left(\sigma(t,x)W(t)\sigma'(t,x)\frac{\partial^2 V^*(t,x)}{\partial x^2}\right) \\ &\quad \left.+ \left|\frac{\partial M(t,x)}{\partial x}\right|^2_{\sigma(t,x)W(t)\sigma'(t,x)}\right]\end{aligned} \tag{50}$$

with boundary condition $V^*(t_F, x) = 0$.

Theorem 5.2 *Assume that the conditions of Theorem 4.4 and Theorem 5.1 are satisfied. Then a nonlinear optimal MCV control law is of the form*

$$k^*_{V|M}(t,x) = \frac{-\beta(x)R^{-1}(t)B'(t)\left(\frac{\partial V^*(t,x)}{\partial x}\right)}{\left|B'(t)\frac{\partial V^*(t,x)}{\partial x}\right|_{R^{-1}(t)}} - \frac{1}{2}R^{-1}(t)B'(t)\left(\frac{\partial M(t,x)}{\partial x}\right), \tag{51}$$

provided that $B'(t)\frac{\partial V^*(t,x)}{\partial x}$ is nonzero, and the optimal cost function V^* satisfies the partial differential equation

$$\begin{aligned}-\frac{\partial V^*}{\partial t} &= -\beta\left|(H^{-1})'B'\frac{\partial V^*}{\partial x}\right| + g'\frac{\partial V^*}{\partial x} - \frac{1}{2}\left(\frac{\partial M}{\partial x}\right)' BR^{-1}B'\left(\frac{\partial V^*}{\partial x}\right) \\ &\quad + \frac{1}{2}\ tr\ \left(EWE'\frac{\partial^2 V^*}{\partial x^2}\right) + \left|\frac{\partial M}{\partial x}\right|^2_{EWE'},\end{aligned} \tag{52}$$

with boundary condition $V^*(t_F, x) = 0$. *Moreover, whenever* $B'(t)\frac{\partial V^*(t,x)}{\partial x}$ *is zero, then* $\beta(x)$ *is also zero; and* $k^*_{V|M}(t, x)$ *employs only the second term* $-\frac{1}{2}R^{-1}(t)B'(t)\left(\frac{\partial M(t,x)}{\partial x}\right)$ *in (51).*

Remark: When $B'(t)\frac{\partial V^*(t,x)}{\partial x}$ is zero, then, M satisfies the partial differential equation for the minimum mean cost problem, as can be seen from the form for $\beta(x)$ in (47). For pairs (t, x) at which this occurs, then, we may see the controller accumulating average cost at the same rate as would a minimum average cost controller.

Proof: From the HJB equation (50), we obtain

$$-\frac{\partial V^*}{\partial t} = \min_{k \in K_M}\left[(g' + k'B')\frac{\partial V^*}{\partial x}\right] + \frac{1}{2}tr\left(EWE'\frac{\partial^2 V^*}{\partial x^2}\right) + \left|\frac{\partial M}{\partial x}\right|^2_{EWE'} \tag{53}$$

with boundary condition $V^*(t_F, x) = 0$. Substituting (46) into (53), we obtain

$$\begin{aligned}-\frac{\partial V^*}{\partial t} &= \min_{|a|=1}\left[\beta a'(H^{-1})'B'\left(\frac{\partial V^*}{\partial x}\right)\right] + g'\frac{\partial V^*}{\partial x} - \frac{1}{2}\left(\frac{\partial M}{\partial x}\right)' BR^{-1}B'\left(\frac{\partial V^*}{\partial x}\right)\\ &\quad + \frac{1}{2}tr\left(EWE'\frac{\partial^2 V^*}{\partial x^2}\right) + \left|\frac{\partial M}{\partial x}\right|^2_{EWE'}.\end{aligned} \tag{54}$$

The minimization is achieved by choosing

$$a = -a^*\left((H^{-1})'B'\left(\frac{\partial V^*}{\partial x}\right)\right), \tag{55}$$

where $a^*\left((H^{-1})'B'\left(\frac{\partial V^*}{\partial x}\right)\right)$ is a unit vector in the direction of $(H^{-1})'B'\left(\frac{\partial V^*}{\partial x}\right)$, when it is nonzero. Notice that, when this quantity is zero, the bracketed term in (54) vanishes. Therefore, we can rewrite (54) in the manner (51), and the result follows. □

Our next step is to examine $\beta(x)$ in the case in which the dynamical system is linear and the cost function accumulates at a quadratic rate. We do not yet assume that the controller is linear, but we shall assume that the average cost function is quadratic:

$$M(t, x) = x'\mathcal{M}(t)x + m(t), \tag{56}$$

so that we can obtain the explicit evaluations

$$\frac{\partial^2 M(t,x)}{\partial x^2} = 2\mathcal{M}(t), \quad \frac{\partial M(t,x)}{\partial x} = 2\mathcal{M}(t)x, \quad \frac{\partial M(t,x)}{\partial t} = x'\dot{\mathcal{M}}(t)x + \dot{m}(t), \tag{57}$$

with which one obtains the following lemma.

Lemma 5.4 *Invoke the assumptions of Lemma 5.3, together with (56). Then, for the quadratic cost rate accumulation and linear dynamical system cases,*

$$\begin{aligned} h(t,x) &= x'(t)Q(t)x(t), \quad Q(t) \geq 0, \\ g(t,x) &= A(t)x(t), \end{aligned} \tag{58}$$

of (35) and (37), we find $\beta(x)$ in (46) to be $\beta(x) = |x|_{\mathcal{R}} - \dot{m} - \text{ tr } (EWE'\mathcal{M})$, where

$$\mathcal{R} \stackrel{\triangle}{=} \mathcal{M}BR^{-1}B'\mathcal{M} - \dot{\mathcal{M}} - Q - A'\mathcal{M} + \mathcal{M}A. \tag{59}$$

Moreover, if $B'(t)\frac{\partial V^*(t,x)}{\partial x}$ is zero when x is zero, then

$$\dot{m} = - \text{ tr } (EWE'\mathcal{M}). \tag{60}$$

A particular case of this situation occurs when

$$V^*(t,x) = x'\mathcal{V}(t)x + v(t), \tag{61}$$

in which case the optimal MCV control law (51) can be rewritten as

$$k^*_{V|M}(t,x) = \frac{-|x|_{\mathcal{R}}R^{-1}(t)B'(t)\mathcal{V}(t)x}{|B'(t)\mathcal{V}(t)x|_{R^{-1}(t)}} - R^{-1}(t)B'(t)\mathcal{M}(t)x, \tag{62}$$

provided that $B'(t)\mathcal{V}(t)x$ is nonzero. When $B'(t)\mathcal{V}(t)x$ is zero, $|x|_{\mathcal{R}}$ is required to vanish. so that only the second term in the right member remains.

Proof: We must return to (47) to find $\beta(x)$, which is rewritten as

$$\beta^2 = \frac{1}{4}\left|B'\frac{\partial M}{\partial x}\right|^2_{R^{-1}} - \frac{\partial M}{\partial t} - \frac{1}{2}tr\left(EWE'\frac{\partial^2 M}{\partial x^2}\right) - h - g'\frac{\partial M}{\partial x} \tag{63}$$

Now substitute (56) to obtain

$$\beta^2 = \frac{1}{4}\left|B'2\mathcal{M}x\right|^2_{R^{-1}} - x'\dot{\mathcal{M}}x - \dot{m} - tr\,(EWE'\mathcal{M}) - h - g'2\mathcal{M}x. \tag{64}$$

From (64) and (58) we obtain

$$\beta^2 = |x|^2_{\mathcal{M}BR^{-1}B'\mathcal{M}-\dot{\mathcal{M}}-Q-(A'\mathcal{M}+\mathcal{M}A)} - \dot{m} - tr\,(EWE'\mathcal{M})\,, \tag{65}$$

which is written as $\beta^2 = |x|^2_{\mathcal{R}} - \dot{m} - tr\,(EWE'\mathcal{M})$. Then, if $\beta(x)$ vanishes when $B'(t)\frac{\partial V^*(t,x)}{\partial x}$ is zero, we have $\dot{m} = -tr\,(EWE'\mathcal{M})$ as a constraint on our choice of $m(t)$. □

Let us now restrict the class of controllers, K_M, to be vector space morphisms. To denote this, we replace the notation K_M by K_{ML}. It follows from the work of Liberty and Hartwig [16] that M and V are then quadratic, which is consistent with the assumptions and results in the foregoing lemma. It is straightforward to see that (62) defines a homogeneous mapping, by $k^*_{V|M}(t, \alpha x) = \alpha k^*_{V|M}(t, x)$. Indeed, the result follows by the definition

$$f(x) = |x|_{\mathcal{R}}/|x|_{\mathcal{V}BR^{-1}B'\mathcal{V}}, \tag{66}$$

on the domain in which $|x|_{\mathcal{V}BR^{-1}B'\mathcal{V}}$ does not vanish, together with the observation that $f(\alpha x) = \alpha f(x)$ on this domain. We shall extend this to the point $x = 0$ momentarily. But the question of whether or not $k^*_{V|M}(t, x)$ is a morphism under the addition of vectors needs further examination. The basic idea which we require is given in the following lemma.

Lemma 5.5 *Let $f(x)$ be given by (66), and consider the controller term $-f(x)R^{-1}B'\mathcal{V}$. If this term is a morphism of vector addition, then $f(x)$ is constant for all x such that $B'\mathcal{V}x$ is nonzero.*

Proof: Denote by x_1 and x_2 two values of x satisfying the assumptions of the lemma. Examine first the case in which x_1 and x_2 are chosen so that $B'\mathcal{V}x_1$ and $B'\mathcal{V}x_2$ are linearly independent. Additivity implies that

$$-R^{-1}B'\mathcal{V}[x_1 + x_2]f(x_1 + x_2) = -R^{-1}B'\mathcal{V}x_1 f(x_1) - R^{-1}B'\mathcal{V}x_2 f(x_2) \tag{67}$$

so that

$$[f(x_1 + x_2) - f(x_1)]B'\mathcal{V}x_1 + [f(x_1 + x_2) - f(x_2)]B'\mathcal{V}x_2 = 0, \tag{68}$$

and it is necessary to conclude that $f(x_1 + x_2) = f(x_1) = f(x_2)$. Turn now to the case in which the two vectors $B'\mathcal{V}x_1$ and $B'\mathcal{V}x_2$ are not linearly independent. Let α be a real number such that $B'\mathcal{V}x_1 = \alpha B'\mathcal{V}x_2$. In this situation, $B'\mathcal{V}(x_1 - \alpha x_2) = 0$. By additivity, it follows that

$$0 = -R^{-1}B'\mathcal{V}x_1 f(x_1) + \alpha R^{-1}B'\mathcal{V}x_2 f(x_2) = -[f(x_1) - f(x_2)]R^{-1}B'\mathcal{V}x_1 \tag{69}$$

so that $f(x_1) = f(x_2)$. □

With the preceding results in hand, we are now able to conclude how the constant character of $f(x)$ induces a corresponding relationship between the weighting matrices in $f(x)$.

Lemma 5.6 *Let $\mathcal{R}$ and $\mathcal{V}BR^{-1}B'\mathcal{V}$ have identical null spaces, and consider the function $f(x)$ defined by (66) on the domain in which $|x|_{\mathcal{V}BR^{-1}B'\mathcal{V}}$ does not vanish. Then $f(x)$ is equal to a (positive) constant γ on this domain, if and only if $\mathcal{R} = \gamma^2\mathcal{V}BR^{-1}B'\mathcal{V}$ on the domain.*

Remark: The assumption that $\mathcal{R}$ has the same null space as $\mathcal{V}BR^{-1}B'\mathcal{V}$ is quite natural. Of course, the null space of $\mathcal{R}$ must contain that of $R^{-1}B'\mathcal{V}$, by our foregoing discussions. Moreover, if this containment were strict, then the only possible constant behavior of our function would be zero. This is of course unduly restrictive. So we set the two null spaces equal to each other.

Proof: Notice that $\mathcal{R}$ and $\mathcal{V}BR^{-1}B'\mathcal{V}$ are nonnegative semidefinite. Thus γ must be seen as positive. Sufficiency of the lemma is of course straightforward. Necessity follows by construction of the gradient vector, which is given by

$$[(f(x))^{-1}\mathcal{R}x - f(x)\mathcal{V}BR^{-1}B'\mathcal{V}x]/|x|^2_{\mathcal{V}BR^{-1}B'\mathcal{V}}, \tag{70}$$

from which it follows that $[\gamma^{-1}\mathcal{R}x - \gamma\mathcal{V}BR^{-1}B'\mathcal{V}x] = 0$ on the domain, and a simple multiplication by γ achieves the result, as desired. Notice that $\mathcal{R}x$ does not vanish on the domain, so that the function f is nonzero for the values of x considered. $\square$

Remark: Inasmuch as this equation holds over the entire nonzero images of the two members, there is clearly no loss of generality in extending the domain of the equation to the whole space.

The solution to the full-state-feedback MCV control problem with linear controller is then given by the following theorem.

Theorem 5.3 *Assume $V^* \in C_p^{1,2}(Q) \cap C(\bar{Q})$ and the same assumptions as in Theorem 5.2 and Lemma 5.4. Then for $k \in K_{ML}$, there exists a linear MCV controller, if and only if there exist solutions M and V to the pair of matrix differential equations*

$$\dot{\mathcal{M}} + A'\mathcal{M} + \mathcal{M}A + Q - \mathcal{M}BR^{-1}B'\mathcal{M} + \gamma^2\mathcal{V}BR^{-1}B'\mathcal{V} = 0, \tag{71}$$

$$\dot{\mathcal{V}} + 4\mathcal{M}EWE'\mathcal{M} + A'\mathcal{V} + \mathcal{V}A - \mathcal{M}BR^{-1}B'\mathcal{V} - \mathcal{V}BR^{-1}B'\mathcal{M} - 2\gamma\mathcal{V}BR^{-1}B'\mathcal{V} = 0, \tag{72}$$

with boundary conditions $\mathcal{M}(t_F) = Q_F$ and $\mathcal{V}(t_F) = 0$, for a suitable positive time function $\gamma(t)$. In such a case, the controller is given by

$$k^*_{V|M}(t, x) = -R^{-1}(t)B'(t)[\mathcal{M}(t) + \gamma(t)\mathcal{V}(t)]x. \tag{73}$$

Remark: In these equations, the choice γ equal to zero results in the classical minimum mean cost situation.

Proof: Using the development above, and (61), (52) is rewritten as

$$\begin{aligned} -|x|^2_{\mathcal{V}} - \dot{v} &= -\sqrt{|x|^2_{\mathcal{R}} - \dot{m} - tr\,(EWE'\mathcal{M})}\ (|2B'\mathcal{V}x|_{R^{-1}}) + tr\,(EWE'\mathcal{V}) \\ &\quad + |2\mathcal{M}x|^2_{EWE'} + 2x'A'\mathcal{V}x - \frac{1}{2}(2\mathcal{M}x)'BR^{-1}B'(2\mathcal{V}x) \end{aligned}$$

with boundary condition $V^*(t_F, x) = 0$. We first note that $\dot{v} = -tr\,(EWE'\mathcal{V})$, $v(t_F) = 0$, so that

$$v(t) = \int_t^{t_F} tr\,(E(\tau)W(\tau)E'(\tau)\mathcal{V}(\tau))\ d\tau.$$

We also have $m(t) = \int_t^{t_F} tr\left(E(\tau)W(\tau)E'(\tau)\mathcal{M}(\tau)\right)\, d\tau$ with $m(t_F) = 0$. Collecting $|x|^2$ terms we end up with

$$0 = |x|^2_{\mathcal{V}+4\mathcal{M}EWE'\mathcal{M}+A'\mathcal{V}+\mathcal{V}A-\mathcal{M}BR^{-1}B'\mathcal{V}-\mathcal{V}BR^{-1}B'\mathcal{M}} - 2|x|_{\mathcal{R}}|x|_{\mathcal{V}BR^{-1}B'\mathcal{V}}, \tag{74}$$

which is denoted by $0 = |x|^2_{\mathcal{E}(t)} - 2|x|_{\mathcal{R}(t)}|x|_{\mathcal{S}(t)}$. Now rewrite (62) in the form

$$k^*_{V|M}(t,x) = -R^{-1}(t)B'(t)\left[\mathcal{M}(t) + \frac{|x|_{\mathcal{R}}(t)}{|x|_{\mathcal{V}(t)B(t)R^{-1}(t)B'(t)\mathcal{V}(t)}}\mathcal{V}(t)\right].$$

Because we are considering only linear controllers, we know from Lemma 5.5 that $f(x)$, defined in that lemma, must be constant on the domain in which $B'(t)\mathcal{V}x$ is nonzero. But then the proof of Lemma 5.6, under our assumptions, implies that $f(x)$ must then be constant for all x, provided that we agree to define it equal to the same constant value on the nullspaces discussed above. Then Lemma 5.6 gives (71). Moreover, (72) is then a consequence of (74). We can then write the expression for the controller in the stated form .

6 Earthquake Application

A three-degree-of-freedom (3DOF) structure under seismic excitation studied in this section. We illustrate how cost mean and minimal cost variance are related to constant parameters γ. Performance characteristics, such as the standard deviation of displacement and control force are also illustrated. Finally, we indicate the way in which control energy is related to the selected structure energy, through the cost functional, as the parameter γ varies.

Consider the 3DOF, single-bay structure with an active tendon controller as shown in Figure 2. The structure is subject to a one-dimensional earthquake excitation. If we assume a simple shear frame model for the structure, then we can write the governing equations of

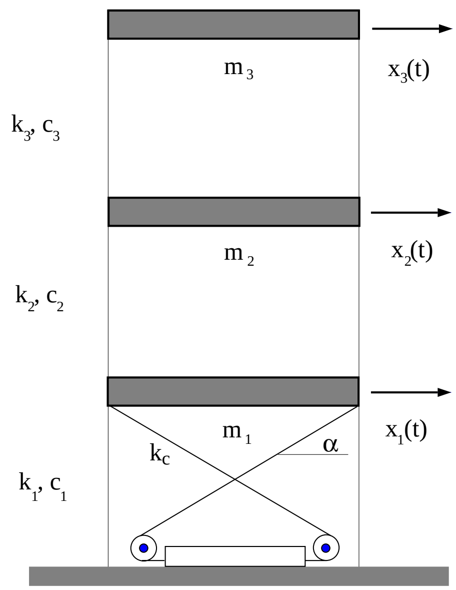

Figure 2: Schematic Diagram for Three Degree-of-Freedom Structure

motion in state space form as

$$dx(t) = \begin{bmatrix} 0 & I \\ -M_s^{-1}K_s & -M_s^{-1}C_s \end{bmatrix} x(t)\ dt + \begin{bmatrix} 0 \\ M_s^{-1}B_s \end{bmatrix} u(t)\ dt + \begin{bmatrix} 0 \\ -\Gamma_s \end{bmatrix} dw(t)$$

where

$$M_s = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad B_s = \begin{bmatrix} -4k_c \cos\alpha \\ 0 \\ 0 \end{bmatrix},$$

$$C_s = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}, \quad \Gamma_s = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad K_s = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix},$$

m_i, c_i, k_i are the mass, damping, and stiffness, respectively, associated with the i-th floor of the building, and k_c is the stiffness of the tendon. The (non-standard) Brownian motion term has $W = 1.00 \times 2\pi$ in^2/sec^3. The parameters were chosen to match modal frequencies and dampings of the experimental structure in [4]. The cost funtion is given by

$$J = \int_0^{t_F} \left(z'(t) K_s z(t) + k_c u^2(t) \right)\ dt,$$

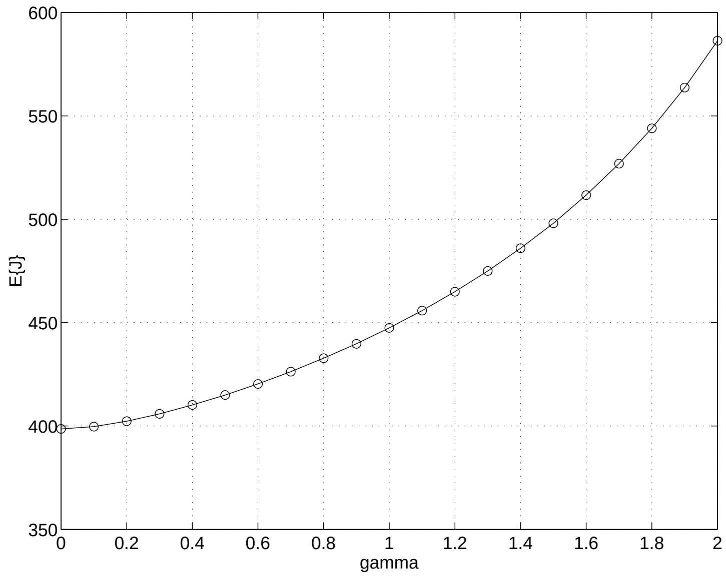

Figure 3: Cost Mean; Full-State-Feedback, MCV, 3DOF

where z is a vector of floor displacements and $x = [z\dot{z}]'$.

Figure 3 shows that the average value of the cost function $E\{J\}$ increases as the MCV parameter γ increases. On the other hand, the minimal associated variance of the cost function decreases. See Figure 4. Recall that the $\gamma = 0$ point corresponds to the classical LQG case. Figure 5 shows the RMS displacement responses of first (σ_{x_1}), second (σ_{x_2}), and third (σ_{x_3}) floor; and the RMS velocity responses of first (σ_{x_4}), second (σ_{x_5}), and third (σ_{x_6}) floor respectively, versus the MCV parameter, γ. It is important to note that both third floor RMS displacement and velocity responses can be decreased by choosing large γ. For larger γ, note that we require larger control force, which means that more effort is needed to reduce the RMS displacement and velocity responses. See Figure 6.

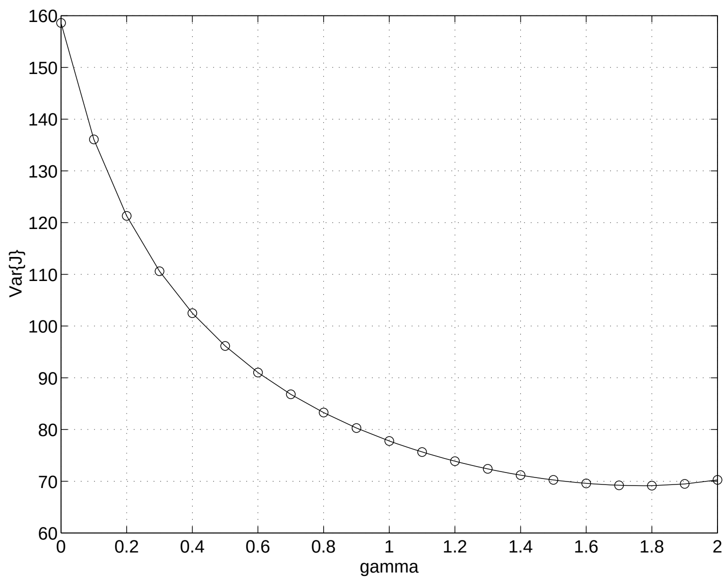

Figure 4: Optimal Variance; Full-State-Feedback, MCV, 3DOF

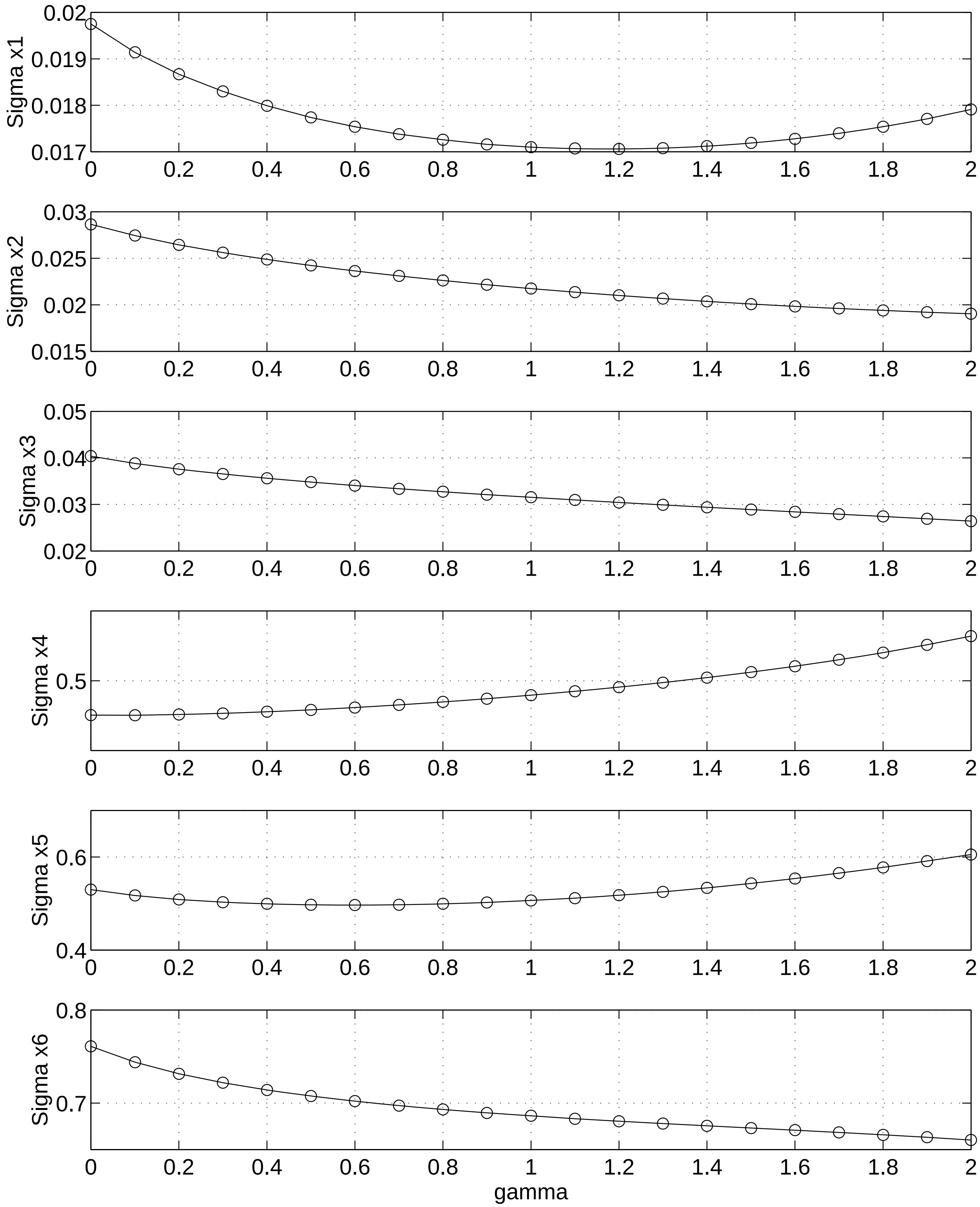

Figure 5: Displacements and Velocities; Full-State-Feedback, MCV, 3DOF

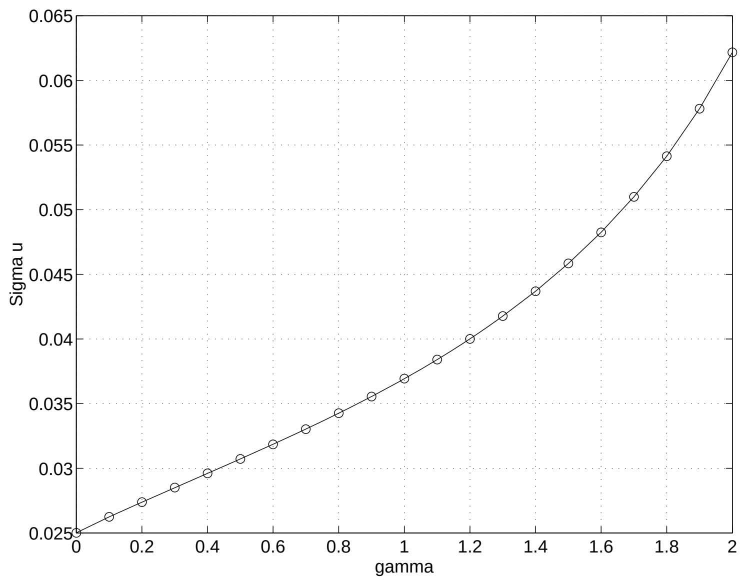

Figure 6: Control Force; Full-State-Feedback, MCV, 3DOF

7 A Simple Distribution Function Shaping Example

Consider a system described on the interval $T = [0, 1]$ by

$$dx(t) = x(t)\ dt + u(t)\ dt + edw(t)$$

where the state $x(t) \in \mathbb{R}$, the control action $u(t) \in \mathbb{R}$, $w(t)$ is a standard Brownian motion with $e^2 = 0.25, x_0 = 1$, and incremental covariance of $x_0 = 0$. The cost function is given by $J = \int_0^1 [x^2(t) + u^2(t)]\ dt$. The MCV feedback controller for the full-state-feedback case is used for several values of γ. For each of these controllers we calculated the feedback control gain matrices and plotted the density and distribution graphs. We also graphed the same items for $J_x = \int_0^1 x^2(t)\ dt$, and $J_u = \int_0^1 u^2(t)\ dt$. Figure 7 shows the density graphs for $\gamma = 0, 2,$ and 4. We note that the mean is smallest for $\gamma = 0$ and largest for $\gamma = 4$. From Figure 8, we notice that the probability of J being smaller than a particular J_0 is largest for $\gamma = 0$ and smallest for $\gamma = 4$ with $\gamma = 2$ in between those two. If we are trying to find a controller that would give the best chance of giving smaller cost, we should choose $\gamma = 0$ in this example.

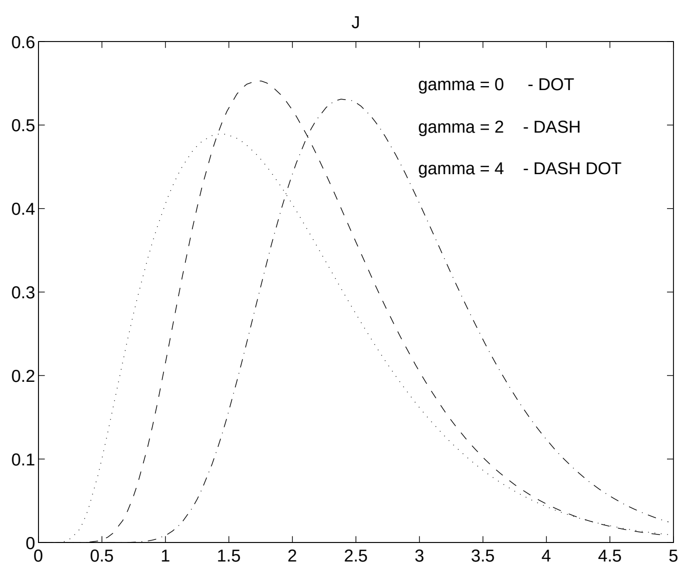

Figure 7: Density Function of the Cost, J; MCV Control

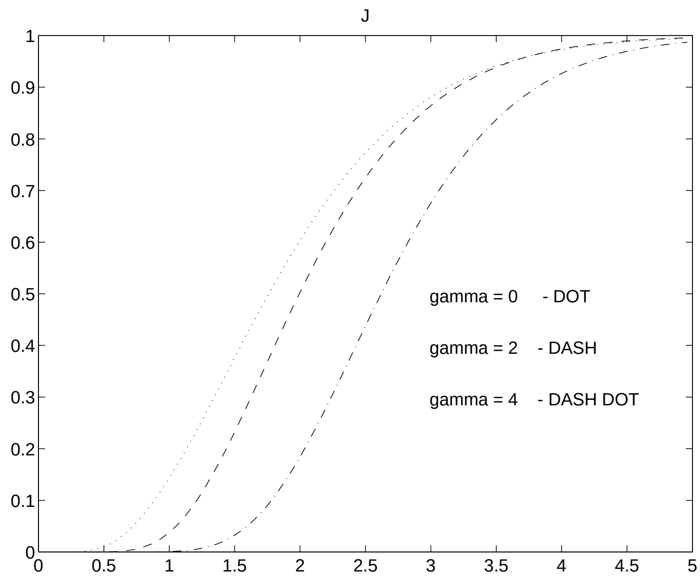

Figure 8: Distribution Function of the Cost, J; MCV Control

In Figures 9 and 10, we have density and distribution graphs for J_x, respectively. Then in Figures 11 and 12, we have density and distribution graphs for J_u. Notice that as γ increases the density of J_x shifts to the left while the J_u graphs shift to the right. This corresponds to the tradeoff between the control effort and the state regulation.

References

[1] T. Başar and P. Bernhard, *H_∞–Optimal Control and Related Minimax Design Problems.* Boston: Birkhäuser, 1991.

[2] A. Bensoussan and J. H. van Schuppen, "Optimal Control of Partially Observable Stochastic Systems with an Exponential-of-Integral Performance Index," *SIAM Journal on Control and Optimization*, Volume 23, pp. 599–613, 1985.

[3] A. Bensoussan, *Stochastic Control of Partially Observable Systems.* London: Cambridge University Press, 1992.

[4] L. L. Chung, T. T. Soong and A. M. Reinhorn, "Experiments on Active Control of MDOF Seismic Structures," *Journal of Engineering Mechanics, ASCE*, Vol. 115, No. 8, pp. 1609–1627, 1989.

[5] W. B. Davenport, Jr. and W. L. Root, *An Introduction to the Theory of Random Signals and Noise.* York: McGraw-Hill Book Company, Inc., 1958.

[6] J. L. Doob, *Stochastic Processes.* New York: John Wiley & Sons, Inc., 1953.

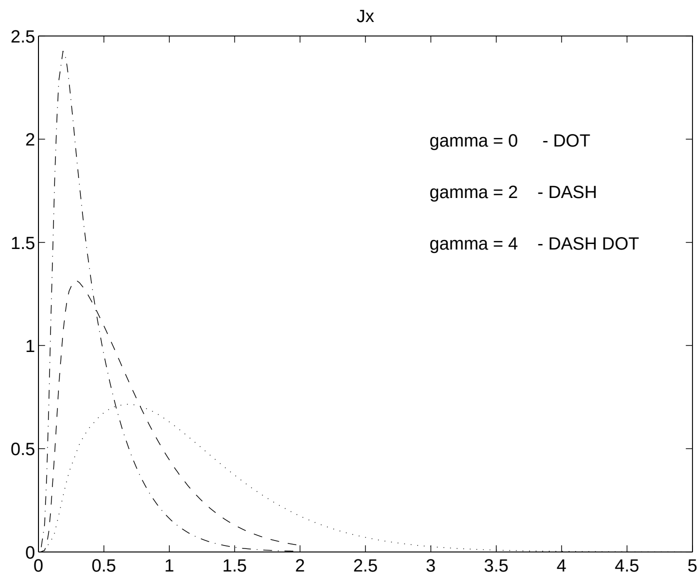

Figure 9: Density Function of the Cost, J_x; MCV Control

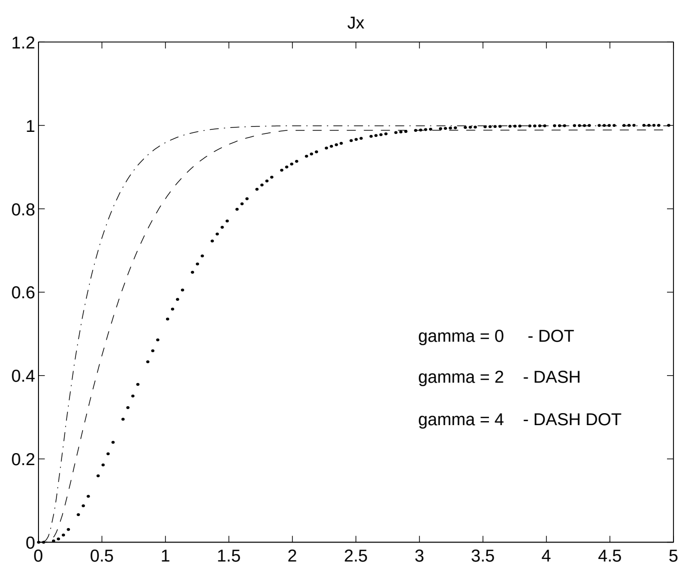

Figure 10: Distribution Function of the Cost, J_x; MCV Control

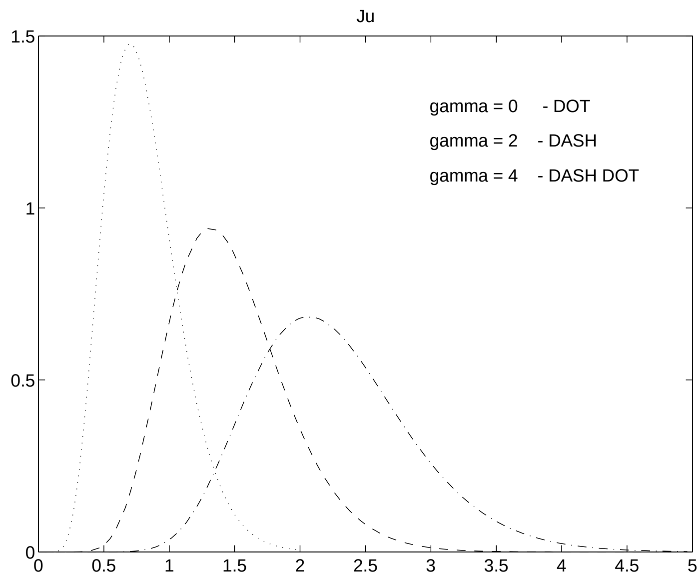

Figure 11: Density Function of the Cost, J_u; MCV Control

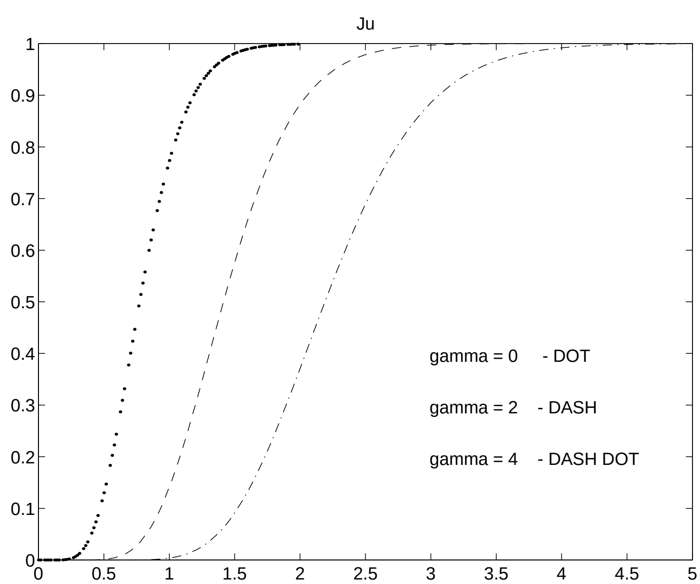

Figure 12: Distribution Function of the Cost, J_u; MCV Control

[7] W. H. Fleming and R. W. Rishel, *Deterministic and Stochastic Optimal Control*. New York: Springer-Verlag, 1975.

[8] W. H. Fleming and W. M. McEneaney, "Risk Sensitive Optimal Control and Differential Games," *Stochastic Theory and Adaptive Control,* in Lecture Notes in Control and Information Sciences 184, T. E. Duncan and B. Pasik-Duncan (Eds.), Springer-Verlag, pp. 185–197, 1992.

[9] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*. New York: Springer-Verlag, 1992.

[10] K. Glover and J. C. Doyle, "State-Space Formulae for All Stabilizing Controllers That Satisfy an H_∞-Norm Bound and Relations to Risk Sensitivity," *Systems and Control Letters*, Volume 11, pp. 167–172, 1988.

[11] K. Glover, "Minimum Entropy and Risk-Sensitive Control: The Continuous Time Case," *Proceedings 28th IEEE Conference on Decision and Control*, pp. 388–391, December 1989.

[12] D. H. Jacobson, "Optimal Stochastic Linear Systems with Exponential Performance Criteria and Their Relationship to Deterministic Differential Games," *IEEE Transactions on Automatic Control*, AC-18, pp. 124–131, 1973.

[13] M. R. James, "Asymptotic Analysis of Nonlinear Stochastic Risk-Sensitive Control and Differential Games," *Mathematics of Control, Signals, and Systems*, Vol. 5, pp. 401–417, 1992.

[14] M. R. James, J. S. Baras, and R. J. Elliott, "Risk-Sensitive Control and Dynamic Games for Partially Observed Discrete-Time Nonlinear Systems," *IEEE Transactions on Automatic Control*, AC-39, No. 4, pp. 780–792, 1994.

[15] P. R. Kumar and J. H. van Schuppen, "On the Optimal Control of Stochastic Systems with an Exponential-of-Integral Performance Index," *Journal of Mathematical Analysis and Applications*, Volume 80, pp. 312–332, 1981.

[16] S. R. Liberty and R. C. Hartwig, "On the Essential Quadratic Nature of LQG Control-Performance Measure Cumulants," *Information and Control*, Volume 32, Number 3, pp. 276–305, 1976.

[17] R. W. Liu and J. Leake, "Inverse Lyapunov Problems," Technical Report No. EE-6510, Department of Electrical Engineering, University of Notre Dame, August 1965.

[18] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkriledze, and E. F. Mischenko, *The Mathematical Theory of Optimal Processes*. New York: Interscience Publishers, 1962.

[19] T. Runolfsson, "The Equivalence between Infinite-horizon Optimal Control of Stochastic Systems with Exponential-of-integral Performance Index and Stochastic Differential Games," *IEEE Transactions on Automatic Control*, Vol. 39, No. 8, pp. 1551–1563, 1994.

[20] Michael K. Sain, "On Minimal-Variance Control of Linear Systems with Quadratic Loss," *Ph.D. Dissertation*, Department of Electrical Engineering, University of Illinois, Urbana, January 1965.

[21] M. K. Sain, "A Sufficiency Condition for Minimum Variance Control of Markov Processes," *Proceedings Fourth Allerton Conf. on Circuit and System Theory*, Monticello, Ill., pp. 593–599, October 1966.

[22] M. K. Sain, "Performance Moment Recursions, with Application to Equalizer Control Laws," *Proceedings Fifth Allerton Conference on Circuit and System Theory*, pp. 327–336, 1967.

[23] M. K. Sain and S. R. Liberty, "Performance Measure Densities for a Class of LQG Control Systems," *IEEE Transactions on Automatic Control*, AC-16, No. 5, pp. 431–439, October 1971.

[24] M. K. Sain, C.-H. Won, and B. F. Spencer, Jr. "Cumulant Minimization and Robust Control," *Stochastic Theory and Adaptive Control,* in Lecture Notes in Control and Information Sciences 184, T. E. Duncan and B. Pasik-Duncan (Eds.), Springer-Verlag, pp. 411-425, 1992.

[25] J. L. Speyer, J. Deyst, and D. H. Jacobson, "Optimization of Stochastic Linear Systems with Additive Measurement and Process Noise Using Exponential Performance Criteria," *IEEE Transactions on Automatic Control*, AC-19, No. 4, pp. 358–366, August 1974.

[26] J. L. Speyer, "An Adaptive Terminal Guidance Scheme Based on an Exponential Cost Criterion with Application to Homing Missile Guidance," *IEEE Transactions on Automatic Control*, AC-21, pp. 371–375, 1976.

[27] K. Uchida and M. Fujita, "On the Central Controller: Characterizations via Differential Games and LEQG Control Problems," *Systems & Control Letters*, Vol. 13, pp. 9-13, 1989.

[28] R. L. Wheeden and A. Zygmund, *Measure and Integral, An Introduction to Real Analysis*. New York: Marcel Dekker, 1977.

[29] P. Whittle, "Risk-Sensitive Linear/Quadratic/Gaussian Control," *Advances in Applied Probability*, Vol. 13, pp. 764–777, 1981.

[30] P. Whittle, *Risk Sensitive Optimal Control*. New York: John Wiley & Sons, 1990.

[31] P. Whittle, "A Risk-Sensitive Maximum Principle: The Case of Imperfect State Observation," *IEEE Transactions on Automatic Control*, Vol. 36, No. 7, pp. 793–801, July 1991.

[32] Chang-Hee Won, Michael K. Sain, and B. F. Spencer, Jr., "Performance and Stability Characteristics of Risk-Sensitive Controlled Structures under Seismic Disturbances,' *Proceedings American Control Conference*, pp. 1926-1930, June 1995

[33] W. M. Wonham, "Stochastic Problems in Optimal Control," *1963 IEEE Int.Conv.Rec.*, part 2, pp. 114–124, 1963.