Satellite Attitude Control Using Statistical Game Theory

Jong-Ha Lee, Ronald W. Diersing and Chang-Hee Won

Abstract—We consider the application of multi-objective statistical game theory to a remote sensing satellite attitude control. Statistical game theory is a generalization of mixed $H_2/H_\infty$ control, where we have two objective functions and we optimize the higher order cumulants of these objective functions. We use previously developed satellite attitude model with thrusters, gravity torquers and a reaction wheel cluster. Then we control the satellite attitude using the statistical game (Minimal Cost Variance/$H_\infty$) control, $H_\infty$ control, and mixed $H_2/H_\infty$ control. Throughout the simulations, statistical game control has an extra degree of freedom to improve the performance and reduce the overshoot and undershoot compared to either $H_\infty$ control and $H_2/H_\infty$ control. The simulations show that the performance of Minimal Cost Variance/$H_\infty$ is 8.8% and 54% faster than $H_2/H_\infty$ and $H_\infty$ control, respectively. Moreover, the control actions of MCV/$H_\infty$ is reduced by 67% compared to $H_\infty$ control and 55% compared to $H_2/H_\infty$ control. So, we achieve both performance improvement while saving the control energy. In the case of the stability margin, MCV/$H_\infty$ control has the highest stability margin, $H_\infty$ control has the lowest, and $H_2/H_\infty$ control an intermediate value between those two.

I. INTRODUCTION

In the stochastic optimal control problem, the system is given as a stochastic differential equation and an optimal controller is determined to minimize the expected values of a cost function. Historically, stochastic optimal control was based upon the mean of a cost function. However, the mean or the first cumulant is only one of the cumulants that describe the distribution of a random variable. Other statistical quantities, such as the variance or the skewness can be considered to minimize the whole distribution of the cost function. This is the main idea of statistical control. In statistical control, a performance index is minimized that is based upon other statistical quantities such as the variance and skewness of the cost function, not just the mean [1]. Recently, we have applied the method to structural control and showed promising results [2].

Multi-objective control is a control method in which the control must concern itself with not only one performance index, as in traditional optimal control, but several. The most prevalent multi-objective control is the mixed $H_2/H_\infty$ control, which the control wishes to minimize an $H_2$ norm while keeping the $H_\infty$ norm constrained. This approach was started in [3], and the Nash game approaches to this problem are proposed in [4]. In this game, two players; a control and a disturbance are considered. They both wish to minimize their respective performance indices when the other player has played their equilibrium solution [5]. In the stochastic version of this problem, the players then wish to minimize the mean of their cost functions.

Minimal Cost Variance (MCV) control minimizes the variance, instead of the mean, of the cost function. The work was started in [6]. The full state feedback MCV control results for a nonlinear system and nonquadratic cost function are given in [7], and a linear system and quadratic cost function case is developed in [8]. The work of Diersing and Sain [9] examined to combine cost cumulants and MCV control to minimize a linear combination of the first two cumulants the mean and the variance while satisfying $H_\infty$ constraint. This is called MCV/$H_\infty$ control and it shows that these control methods have been applied successfully to vibration control problems, such as the control structures excited by seismic disturbance [9].

In this paper, multi-objective statistical control is used for a Low Earth Orbit (LEO) satellite. For this, we consider MCV/$H_\infty$ control with actual commercial satellite parameters. The work of [11] compared mixed $H_2/H_\infty$ control with $H_2$ control, and $H_\infty$ control method to stabilize attitude control. This comparison was performed for a one player system. Here, we consider a stochastic version of the system with quadratic cost function. When the system has a stochastic white noise present in addition to a bounded power disturbance, the players’ performance indices can be cast in terms of the mean value of the two quadratic costs. In a Nash game approach, a control and a disturbance is taken as two players. The equilibrium solution is found by solving coupled Riccati equations.

We present multi-objective statistical control method with equilibrium solutions in Section II. In Section IV, Korea Multi Purpose Satellite (KOMPSAT) system model with three magnetic torquers and a four reaction wheel cluster as actuators is provided. In section V, the performance of two control, $H_\infty$ and $H_2/H_\infty$ is compared to that of MCV/$H_\infty$ control with real parameters of a LEO satellite. Finally, Section VI concludes this paper.

II. MULTI-OBJECTIVE STATISTICAL CONTROL

The system is described as a linear system

$$dx(t) = [Ax(t) + Bu(t) + Dv(t)]dt + Ed\xi(t)$$

(1)

where $x$ is the state, $u$ is the control, $v$ is the disturbance, and $\xi$ is a Brownian motion with variance $W$. The regulated
outputs of the system are
\[ z_1(t) = H_1(t)x(t) + G_1(t)u(t) \]  
(2)
\[ z_2(t) = H_2(t)x(t) + G_2(t)u(t) \]  
(3)
We assume that \( H_i' H_i = Q_i \), and \( G_i' G_i = R_i \) for \( i = 1,2 \), where \( Q_1, Q_2, \) and \( R_2 \) are assumed positive semi-definite and \( R_1 \) is assumed positive definite. This stochastic differential equation has two cost functions. The first cost function, \( J_1 \), is to be associated with the control \( u \) and the second, \( J_2 \) is for the disturbance \( v \). The players’ cost functions will be assumed to be quadratic:
\[
J_1 = \int_{t_0}^{t_f} (z_1'(t)z_1(t)) \, dt 
\]
(4)
\[
J_2 = \int_{t_0}^{t_f} \left( \delta^2 v'(t)v_1 x(t) - z_2'(t)z_2(t) \right) \, dt 
\]
(5)
With the system and costs defined, we can discuss the control method. The game to be considered here is one in which the first player, the control \( u \), wishes to minimize a performance index consisting of a linear combination of cumulants given by \( \phi_1 = E_{t_0} \gamma_1 + \gamma V \), where \( \gamma \) is some positive constant and \( V \) is the variance using the conditional expectation. On the other hand, the second player, the disturbance \( v \), wishes to minimize the mean of its cost function. That is the disturbance has \( \phi_2 = E_{t_0} \gamma_2 \) as its performance index. Because both players will be assumed to have feedback information available to them, \( U_M \) will be the information pattern for the control and \( V_\phi \) will be the information pattern for the disturbance. We call this MCV/H\(_\infty\) control. We assume that the costs are quadratic. That is \( M(t,x) = x' M(t)x + m(t), V(t,x) = x' V(t)x + v(t) \), and \( P(t,x) = x' P(t)x + p(t) \) where \( M, V, P \) are matrix functions of time and \( m, v, p \) are scalar functions of time. The MCV/H\(_\infty\) equilibrium solution for the control is determined as
\[
\min_{\mu \in U_M} \{ x' M x + \bar{m} + 2(Ax + B \mu + Dv*)' M x \\
+ (EWE'(M) + \gamma V)x + \bar{x}' Q_1 x + \bar{\mu}' R_1 \mu \\
+ \gamma [x' \bar{\nu} x + \bar{v} + 2(Ax + B \mu + Dv*)' \bar{\nu} x] \\
+ 4 \gamma V \bar{\nu} E' M \mu \} = 0
\]
and minimizing this gives
\[
u^* = \mu^*(t,x) = - R^{-1} B' [M(t) + \gamma V(t)]x(t)
\]
(7)
Similarly, for the disturbance
\[
\min_{v \in V_\phi} \{ x' \bar{P} x + \bar{p} + 2(Ax + B \mu^* + Dv)' \bar{P} x \\
+ (EWE'(P)) + \gamma^2 v' v - x' Q_2 x \\
- \mu' R_2 \mu^* \} = 0
\]
(8)
which by minimization yields
\[
v^* = \nu^*(t,x) = - \frac{1}{\delta^2} D' D \bar{P} t x(t)
\]
(9)
Using this equilibrium solution \( (\mu^*, \nu^*) \), we can determine three Riccati equations by substitution. \( M, V \) and \( P \) are solutions of these three Riccati equations. There is the mean of the control’s cost function
\[
\dot{M} + A'M - MA + Q_1 - MBR_1^{-1} B'M = 0
\]
(10)
where \( M(t_f) = Q_f \). Next, we derive an expression for the variance.
\[
\dot{V} + A'V - VA - \gamma MB R_1^{-1} B'V - \gamma V BR_1^{-1} B'M
\]
\[
- \frac{1}{\delta^2} PDD'P - \frac{1}{\delta^2} VDD'P - 2\gamma VBR_1^{-1} B'V
\]
\[
+ 4 MEW E' M = 0
\]
with \( \nu(t_f) = 0 \). Finally an expression for the mean of the disturbance’s cost is given by
\[
\dot{\bar{P}} + A' \bar{P} + \bar{P} A - (M + \gamma V)BR_1^{-1} B' \bar{P}
\]
\[
- \bar{P}BR_1^{-1} B'(M + \gamma V) - \frac{1}{\delta^2} PDD'P
\]
\[
- Q_2 - MB R_1^{-1} R_2 R_1^{-1} B'M
\]
\[
- \gamma MB R_1^{-1} R_2 R_1^{-1} B'V - \gamma VBR_1^{-1} R_2 R_1^{-1} B'M
\]
\[
- \gamma^2 VBR_1^{-1} R_2 R_1^{-1} B'V = 0
\]
(12)
with \( P(t_f) = Q_f \). Notice that when the system is linear and the costs are quadratic in the stochastic game, we know the equilibrium solution \( \mu^*, \nu^* \) when these Riccati equations are satisfied [2]. Also as \( \gamma \) goes to zero, the equilibrium solution becomes the solution of \( H_2/H_\infty \) control problem. This suggests that MCV/H\(_\infty\) control is a cumulant generalization of \( H_2/H_\infty \) control [9].

III. MCV/H\(_\infty\) Control as a Generalization of \( H_2/H_\infty \) Control

In the \( H_2/H_\infty \) control technique, one wishes to minimize the \( H_2 \) norm of the system while constraining the \( H_\infty \) norm to some value, in our case we will call this value \( \delta \). The Nash game approach to solving the \( H_2/H_\infty \) problem was undertaken by [4] and [10]. The \( H_2/H_\infty \) control problem can be approached by finding a Nash equilibrium solution to a two person game in which one player is the control and the other player is the disturbance. For the stochastic case, the control wishes to minimize the mean of its cost function \( J_1 \) and the disturbance wishes to minimize the mean of \( J_2 \), the disturbance’s cost function as given in (5). The control wishing to minimize the mean of \( J_2 \) corresponds with the \( H_2 \) norm, while the mean of \( J_2 \) corresponds with a constraint on the \( H_\infty \) norm. The stochastic differential equation governing the system is still (1). For \( E\{J_2\} \geq 0 \), we have
\[
E \left\{ \int_{t_0}^{t_f} (\delta^2 v'(t)v(t) - z_2'(t)z(t)) \, dt \right\} \geq 0.
\]
(13)
With some manipulation this becomes
\[
E \left\{ \int_{t_0}^{t_f} \delta^2 v'(t) v(t) \, dt \right\} \geq E \left\{ \int_{t_0}^{t_f} \dot{z}_2(t) z(t) \, dt \right\}
\]
and by interchanging the integration and expectation, we obtain
\[
\int_{t_0}^{t_f} \Delta^2 E \{ v'(t) v(t) \} \, dt \geq \int_{t_0}^{t_f} E \{ \dot{z}_2(t) z(t) \} \, dt.
\]
This then becomes
\[
\int_{t_0}^{t_f} E \left\{ \| z_2(t) \|^2 \right\} \, dt \leq \Delta^2 \int_{t_0}^{t_f} E \left\{ \| v(t) \|^2 \right\} \, dt \tag{14}
\]
where \( \| z(t) \|^2 = z'(t) z(t) \).
However, the two-norm for the stochastic case and signal \( w \), as given in [10], is defined as
\[
\| w \|^2[t_0, t_f] = \int_{t_0}^{t_f} E \left\{ \| w(t) \|^2 \right\} \, dt. \tag{15}
\]
Using this and substituting into (14), we have
\[
\| z_2(t) \|^2[t_0, t_f] \leq \| v(t) \|^2[t_0, t_f]. \tag{16}
\]
The \( H_\infty \) norm of a system \( T_{2v} \) is defined as
\[
\| T_{2v} \|_{H_\infty} = \sup_v \left\{ \| z_2 \|^2[t_0, t_f] \right\}. \tag{17}
\]
What we can see is that by minimizing the mean of \( J_2 \) with respect to the disturbance \( v \), we in effect constrain the \( H_\infty \) norm of the system.

Using the Nash game approach to the \( H_2/H_\infty \) problem gives the following solutions from [4],[10].
\[
\begin{align*}
    u^* &= \mu^*(t, x) = -R^{-1}(t) B(t) M(t) x(t) \\
v^* &= v^*(t, x) = -\frac{1}{\Delta^2} D' P(t) x(t)
\end{align*}
\]
with
\[
\begin{align*}
    \dot{M} + A' M + M A + Q_1 - M B R_1^{-1} B' M \\
    - \frac{1}{\Delta^2} P D D' M - \frac{1}{\Delta^2} M D D' P &= 0
\end{align*}
\]
and
\[
\begin{align*}
    \dot{P} + A' P + P A - M B R_1^{-1} B' P \\
    - P B R_1^{-1} B' M - \frac{1}{\Delta^2} P D D' P &= 0
\end{align*}
\]
\[
\begin{align*}
    - Q_2 - M B R_1^{-1} R_2 R_1^{-1} B' M
\end{align*}
\]
In MCV control, we minimize the variance of \( J_1 \) while holding the mean of \( J_2 \) to a constraint. This can be seen as a generalization of \( H_2 \) control. Meanwhile, we also consider some uncertainty to the system that can be better viewed through the \( H_\infty \) technique. In the same vein, we can view MCV/\( H_\infty \) as a generalization of \( H_2/H_\infty \). Therefore we generalize the Nash game approach to the MCV/\( H_\infty \) case, where the control considers the variance of its cost \( J_1 \) and the disturbance still considers the mean of \( J_2 \). The disturbance’s performance index of the mean of its cost corresponds to a constraint on the \( H_\infty \) norm, as has been shown.

Notice that these Riccati equations (20) are very similar to those in the MCV/\( H_\infty \) case. In fact, as the parameter \( \gamma \) goes toward zero, the MCV/\( H_\infty \) control Riccati equations become these from the \( H_2/H_\infty \) problem. This is a generalization of the \( H_2/H_\infty \) problem. As the parameter \( \gamma \) goes toward zero, the weight of the variance on the performance index becomes less and less, which puts the weight on the mean constraint. When it is in fact zero, the problem is simply the \( H_2/H_\infty \) control problem. So through the use of cumulants and Nash game theory, we can have a generalization of \( H_2/H_\infty \). Furthermore, we can allow the control to use higher order cumulants as in the MCV control problem, but also design for some uncertainties that can be represented through the \( H_\infty \) norm of the system.

IV. System Model

We consider the satellite’s attitude control using four reaction wheels and three thrusters. The actual system description and its parameters of a LEO satellite KOMPSAT is given in [11]. The KOMPSAT is a sun-synchronous remote sensing satellite with the inclination of 98.13 degrees, the altitude of 685 km, and the total weight of 509 kg. The general nonlinear satellite attitude dynamic model is given by
\[
\begin{align*}
    I_g \dot{\omega} &= -\omega \times (I_1 \omega + L' I_1 \Omega) - L' \Omega \omega \\
    &+ \Omega \text{thrust} + \Omega \text{gravity} + \psi
\end{align*}
\]
where \( \psi \) represents the disturbances due to the magnetic field, solar radiation pressure, and atmospheric drag. The states are given as \( x = [\phi, \theta, \psi, \omega_x, \omega_y, \omega_z, \Omega_1, \Omega_2, \Omega_3, \Omega_4]' \equiv [\phi_x, \omega, \Omega]' \) where \( \omega_x, \omega_y, \omega_z \) are the angular velocities in Body Fixed Coordinate (BFC) system, \( \Omega \) are the wheel speeds, \( \phi \) is the roll Euler angle, \( \theta \) is the pitch Euler angle, and \( \psi \) is the yaw Euler angle. We assume that roll, pitch, and yaw angles are available using the conical earth sensor, the fine sun sensor, and the gyros. In addition, the wheel speeds are also available from the measurement of the reaction wheel tachometer. Also the following are defined:

1) \( n \): Orbital rate
2) \( I_1 \): Total amount of inertia for satellite body (3 \times 3)
3) \( I_w \): Wheel moment of inertia matrix (4 \times 4)
4) \( I_g = I_1 - L' I_1 \Omega \): Total moment inertial minus moment of inertia of the wheels (3 \times 3)
5) \( L \): Wheel orientation matrix (4 \times 3)

\[
L = \begin{bmatrix}
    \cos \alpha \sin \beta & \sin \alpha \sin \beta & \cos \beta \\
    -\sin \alpha \sin \beta & \cos \alpha \sin \beta & \cos \beta \\
    -\cos \alpha \sin \beta & -\sin \alpha \sin \beta & \cos \beta \\
    \sin \alpha \sin \beta & -\cos \alpha \sin \beta & \cos \beta
\end{bmatrix}
\]

To have zero initial conditions we let \( \delta \omega_y = \omega_y + n \), and find the general linear equation form of (21) for any Torque Equilibrium Attitude (TEA) of the states. For the simplicity sake, we assume the case of the attitude hold mode where the TEA values are fixed at the following values: \( \omega_y = [0, 0, 0, 0, 0, 0, 0, 0, 0]' \). Assuming the gravity gradient torque for a point mass, we have the following linearized equations.
\[
\dot{\omega} \equiv I_g^{-1} n N_1 \omega + I_g^{-1} n N_2 \Omega + I_g^{-1} 3n^2 N_3 \phi \omega \\
- I_g^{-1} L' \omega \text{wheel} + I_g^{-1} \Omega \text{thrust} + I_g^{-1} \omega \text{gravity}.
\]
and
\[
\dot{\Omega} = I_w^{-1} \tau_w - L\dot{\omega},
\]
(24)
and
\[
\begin{align*}
\dot{\phi} &= [0, 0, -n]'\phi + [n, 0, 0]'\psi + [1, 0, 0]'\omega_x \\
&\quad + [0, 1, 0]'(\omega_y + n) + [0, 0, 1]'\omega_z.
\end{align*}
\]
(25)
Because \(\delta\omega_y = \omega_y + n\), we have
\[
\dot{\phi} = n\psi + \omega_x
\]
\[
\dot{\theta} = \omega_y + n = \delta\omega_y
\]
\[
\dot{\psi} = -n\phi + \omega_z
\]
(26)
Using (23), (24), and (26), we can obtain a linearized differential equation of (1) with
\[
A = \begin{bmatrix}
X_1 & I_3 & 0_{3\times 4} \\
3I_g^{-1}n^2N_3 & I_g^{-1}nN_1 & I_g^{-1}nN_2 \\
-3LI_g^{-1}n^2N_3 & -LI_g^{-1}nN_1 & -LI_g^{-1}nN_2
\end{bmatrix}
\]
(27)
where
\[
X_1 = \begin{bmatrix}
0 & 0 & n \\
0 & 0 & 0 \\
-n & 0 & 0
\end{bmatrix}
\]
(28)
\[
N_1 = \begin{bmatrix}
0 & 0 & I_t(3, 3) - I_t(2, 2) \\
0 & 0 & 0 \\
I_t(2, 2) - I_t(1, 1) & 0 & 0
\end{bmatrix}
\]
(29)
\[
N_2 = \begin{bmatrix}
I_w(1, 1)L_t(3, 1) & I_w(2, 2)L_t(3, 2) & 0 \\
0 & 0 & 0 \\
-I_w(1, 1)L_t(1, 1) & -I_w(2, 2)L_t(1, 2) & 0
\end{bmatrix}
\]
(30)
\[
N_3 = \begin{bmatrix}
I_t(3, 3) - I_t(2, 2) & 0 & 0 \\
0 & I_t(3, 3) - I_t(1, 1) & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
(31)
\(L_t = L_t')\) and \(X(i, j)\) is the element of a matrix \(X\) in the \(i\)th row and \(j\)th column. The control is given by the torque due to reaction wheels and the thrusters, \(\tau = [\tau_w, \tau_{\text{thruster}}]'\).
The control's input matrix is given as
\[
B = \begin{bmatrix}
0_{3 \times 4} & 0_{3 \times 4} \\
-I_g^{-1}L' - I_g^{-1}I_g^{-1}L' & I_g^{-1} \\
I_w + LI_g^{-1}L' - LI_g^{-1}L' & I_g^{-1}
\end{bmatrix}
\]
(32)
and the disturbance's input matrix is
\[
D = \begin{bmatrix}
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\
I_g^{-1} & I_g^{-1} & I_g^{-1} \\
-LI_g^{-1} & -LI_g^{-1} & -LI_g^{-1}
\end{bmatrix}
\]
(33)
and \(E\) is a vector of ones.

### Table I

<table>
<thead>
<tr>
<th>Control Method</th>
<th>(\omega_x)</th>
<th>Time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(H_\infty (\delta = 4))</td>
<td>5.0e-5</td>
<td>128.38</td>
</tr>
<tr>
<td>(H_2/H_\infty (\delta = 4))</td>
<td>5.0e-5</td>
<td>123.77</td>
</tr>
<tr>
<td>MCV/(H_\infty (\delta = 4, \gamma = 1.0e-7))</td>
<td>5.0e-5</td>
<td>117.17</td>
</tr>
</tbody>
</table>

### V. Simulation Results

In this section, we investigate MCV/\(H_\infty\) control performance and compare the angular velocity and the control action as well as the stability margin of \(H_\infty\) and \(H_2/H_\infty\) control. All simulations in this paper are based on the following parameters; orbital rate \(n = 1.06e-3\) rad/s, total moment of inertia for the spacecraft body,
\[
I_t = \begin{bmatrix}
217.38 & 0 & 0 \\
0 & 95.57 & 0 \\
0 & 0 & 154.57
\end{bmatrix}
\]
and the moment of inertia matrix for the reaction wheels,
\[
I_w = 0.0077 \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
We let the state weighting matrix \(Q_1 = Q_2 = Q = I_{10 \times 10}\) and the control weighting matrix \(R_1 = R_2 = R = I_{7 \times 7}\).
The initial condition is given in the simulation as \(x_0 = [1\text{deg}, 1\text{deg}, 1\text{deg}, 0, -n, 0, 0, 0, 0, 0]\).

![Fig. 1. Time when Angular Velocity \(\omega_x\) is 5.0e-5 deg/sec Versus \(\gamma\)](image-url)

Before we start, we have to determine the parameter \(\delta\) of \(H_\infty\), \(H_2/H_\infty\), and MCV/\(H_\infty\) control and the parameter \(\gamma\) of MCV/\(H_\infty\) control. For fixed \(\delta\), we can always find \(\gamma\) which will improve the performance of the system. In the first simulation, we fixed \(\delta = 4\) and vary \(\gamma\) for the performance comparison. We define the settling time as the
control is a generalization of $H_\infty$ that if $\gamma = 1.0e^{-7}$ and $H_\infty$ settling time of $MCV/H_\infty$ control. Notice that if $\delta = 4$, three Riccati equations have unique solutions $M$, $V$, $P$ from $\gamma = 1.0e^{-5.6}$. Since the settling time of $MCV/H_\infty$ control is faster than that of $H_\infty$ and $H_2/H_\infty$ control from $\gamma = 1.0e^{-6.2}$ to $1.0e^{-8}$, we use $\gamma = 1.0e^{-7}$ for the following simulation. Also one can notice that if $\gamma$ of $MCV/H_\infty$ control approaches 0, then we have $H_2/H_\infty$ control. From this, we can verify that $MCV/H_\infty$ control is a generalization of $H_2/H_\infty$ control.

Figure 2 shows the angular velocity $\omega_x$ versus time graph for $H_\infty$, $H_2/H_\infty$, and $MCV/H_\infty$ control. Table I shows the angular velocity $\omega_x$ and the time when $\omega_x$ is 5.0e-5 deg/sec. It shows that the time required to reach 5.0e-5 deg/sec of $\omega_x$ in the $MCV/H_\infty$ control is 117.12 sec which is 6.65 sec quicker than that of $H_2/H_\infty$ control. Table II shows the largest overshoot and undershoot values of Figure 2. $MCV/H_\infty$ control has the smallest overshoot and $H_\infty$ control has the highest overshoot. $H_2/H_\infty$ control performance is between the other two control methods.

The wheel speed $\Omega_1$ is also simulated in Figure 3. In this plot, one can see that $MCV/H_\infty$ control has the smallest undershoot at $-1.1e^{-6}$, and has the fastest settling time. $H_\infty$ control has the largest undershoot at $-5.1e^{-6}$, and has the slowest settling time. The undershoot and the settling time of $H_2/H_\infty$ control is in between $MCV/H_\infty$ and $H_\infty$ control. We note that the other three wheel speeds $\Omega_2$, $\Omega_3$, and $\Omega_4$ achieve the similar performance as $\Omega_1$.

We have two control actions given by reaction wheel torques $\Sigma_{wheel}$ and thruster torques $\Sigma_{thruster}$. Figure 4 plots the magnitude of $\Sigma_{wheel}$ versus time and Table III shows the amount of $\Sigma_{wheel}$ for the specific times. We choose the times to match the times of Table I. At those times, we found that the magnitude of $\Sigma_{wheel}$ value of $MCV/H_\infty$ control is

<table>
<thead>
<tr>
<th>Control Method</th>
<th>Overshoot</th>
<th>Undershoot</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_\infty (\delta = 4)$</td>
<td>6.30e-4</td>
<td>-3.12e-2</td>
</tr>
<tr>
<td>$H_2/H_\infty (\delta = 4)$</td>
<td>1.40e-4</td>
<td>-3.09e-2</td>
</tr>
<tr>
<td>$MCV/H_\infty (\delta = 4$ and $\gamma = 1.0e^{-7})$</td>
<td>5.42e-5</td>
<td>-3.01e-2</td>
</tr>
</tbody>
</table>

**TABLE II**
Overshoot and Undershoot Angular Velocity $\omega_x$ of Three Control Methods

| Control Method | Time | $|\Sigma_{wheel}|$
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_\infty (\delta = 4)$</td>
<td>128.38s</td>
<td>1.3e-10</td>
</tr>
<tr>
<td>$H_2/H_\infty (\delta = 4)$</td>
<td>123.75s</td>
<td>1.4e-10</td>
</tr>
<tr>
<td>$MCV/H_\infty (\delta = 4$ and $\gamma = 1.0e^{-7})$</td>
<td>117.12s</td>
<td>6.3e-11</td>
</tr>
</tbody>
</table>

**TABLE III**
Control Action due to Reaction Wheels for Three Control Methods
the largest and $H_\infty$ control is the largest. We also show the control action due to thrusters $\tau_{\text{thruster}}$ in Figure 5 and Table IV. From these results, we find that MCV/$H_\infty$ control requires the smallest control effort, and $H_\infty$ control requires the largest. Again, $H_2/H_\infty$ is in between two controls.

Finally, we provide the stability results. Table V shows the maximum real part of the closed loop system poles for $H_\infty$, $H_2/H_\infty$, and MCV/$H_\infty$ control. $H_\infty$ control has the largest closed loop eigenvalue, MCV/$H_\infty$ control has the smallest eigenvalue, and $H_2/H_\infty$ is in between. From the stability point of view, MCV/$H_\infty$ control gives the highest stability margin, $H_\infty$ control gives the smallest, and $H_2/H_\infty$ control is in between.

### VI. CONCLUSIONS

In this paper, multi-objective statistical control method for satellite attitude control is studied. In doing this, we consider a stochastic version of a system and a system with two players, a control and a disturbance. A Nash game approach was taken to solve the case when two players wish to optimize their own performance indices, namely two different linear combinations of cost cumulants. The performance comparison of MCV/$H_\infty$ control with two different control methods $H_\infty$ and $H_2/H_\infty$ is provided. We compared these three control methods for $\delta = 4$ and $\gamma = 1.0\times10^{-7}$. By comparing those performances, MCV/$H_\infty$ control has the fastest settling time in reaching 5.0e-5 deg/sec angular velocity $\omega_x$, and requires the least control effort, and $H_\infty$ control has the slowest settling time and requires the most control effort. $H_2/H_\infty$ control showed the intermediate performance between MCV/$H_\infty$ and $H_\infty$ control. MCV/$H_\infty$ control showed the smallest overshoot and undershoot in time response because we optimized with respect to the variations. Finally, from the stability point of view, MCV/$H_\infty$ control has the largest stability margin, $H_\infty$ control has the smallest, and again $H_2/H_\infty$ control is in between. We conclude that MCV/$H_\infty$ control is a cumulant generalization of $H_2/H_\infty$ control and the performance of MCV/$H_\infty$ control is better than $H_\infty$ and $H_2/H_\infty$ control for satellite’s attitude control.

### REFERENCES


